Advanced Combinatorics - 2016 Fall
Solutions to Exercise 1

Comments and corrections are welcome.

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1. Prove that for each $n \in \mathbb{N}$, if G is a triangle free graph with n vertices and $e(G) = \lfloor n^2/4 \rfloor$ then G is isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

   Solution: Suppose n is even, then from the first proof of Mantel's theorem in the class, we see that if the equality holds, then for any $(u, v) \in E$, $d(u) + d(v) = n$. In particular, it implies that $\Gamma(u) \cup \Gamma(v) = V$. Again, since G is triangle-free, there is no edge between vertices in $\Gamma(u)$ or $\Gamma(v)$. We see that $L = \Gamma(u)$ and $R = \Gamma(v)$ form a bipartition of all vertices. Thus, G has to be a complete bipartite graph $K_{s,t}$ to maximize the number of edges, where $s + t = n$.

   Otherwise, n is odd. We want to use the same argument, which only requires *an* edge $(u, v) \in E$, such that $d(u) + d(v) = n$. Suppose otherwise, then for any edge $(u, v) \in E$, $d(u) + d(v) \leq n - 1$. Follow the same proof we see that $e(G) \leq \frac{n(n-1)}{4}$, which is impossible.

   It is easy to see that $e(G) = st$. Given $s+t = n$, to maximize $st = s(n-s) = ns-s^2$, it must be that $s = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$.

2. Show that if G is a graph with n vertices, m edges, and t triangles, then

$$t \geq \frac{m}{3n}(4m - n^2).$$

   Find an infinite family of graphs such that the above equality holds; that is,

$$t = \frac{m}{3n}(4m - n^2).$$

   Solution: For each edge $(u, v) \in E$, let t_{uv} be the number of triangles containing (u, v). Thus,

$$t_{uv} = |\Gamma(u) \cap \Gamma(v)| = |\Gamma(u)| + |\Gamma(v)| - |\Gamma(u) \cup \Gamma(v)| \geq d(u) + d(v) - n,$$

   or equivalently,

$$d(u) + d(v) \leq t_{uv} + n.$$

   Sum over all edges $(u, v) \in E$,

$$\sum_{v \in V} d(v)^2 \leq \sum_{uv \in E} t_{uv} + e(G)n.$$

Note that each triangle is counted three times in $\sum_{uv\in E} t_{uv}$. Hence $\sum_{uv\in E} t_{uv} = 3t$. It implies that

$$\sum_{v\in V} d(v)^2 \leq 3t + mn.$$

Then we apply Cauchy-Schwarz and the Handshaking lemma:

$$\begin{aligned} 3t + mn \geq \sum_{v\in V} d(v)^2 &\geq \frac{\left(\sum_{v\in V} d(v)\right)^2}{n} \\ &= \frac{4m^2}{n}. \end{aligned}$$

Rearranging the terms yields the desired inequality.

For the second part, verify that complete graphs K_n satisfy the requirement ($m = \binom{n}{2}$ and $t = \binom{n}{3}$).

3. For each integer $n \geq 3$, what is the maximum possible number of edges in a graph with n vertices which contains exactly one triangle. Prove your answer.

Solution: The answer is

$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2.$$

We first show that $e(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 2$ for any graph G with exactly one triangle.

Suppose the triangle is $T = \{u, v, w\}$. Then consider $S = G\backslash T$. It is easy to see that

$$e(G) = e(S) + e(S,T) + 3,$$

where $e(S,T)$ is the number of edges between S and T. Clearly S is triangle-free. Hence by Mantel's theorem,

$$e(S) \leq \left\lfloor \frac{(n-3)^2}{4} \right\rfloor.$$

Moreover, $\Gamma(u)$, $\Gamma(v)$, and $\Gamma(w)$ must be disjoint, as otherwise we have another triangle. It implies that

$$e(S,T) = d(u) + d(v) + d(w) \leq v(S) = n - 3.$$

Put everything together:

$$\begin{aligned} e(G) \leq 3 + n - 3 + \left\lfloor \frac{(n-3)^2}{4} \right\rfloor &= n + \left\lfloor \frac{n^2 - 6n + 9}{4} \right\rfloor \\ &= n + \left\lfloor \frac{n^2 - 2n + 1}{4} - n + 2 \right\rfloor \\ &= 2 + \left\lfloor \frac{(n-1)^2}{4} \right\rfloor. \end{aligned}$$

We also need to show that this upper bound can be achieved. We take an instance achieving the upper bound of Mantel's theorem with $n-1$ vertices; namely, $K_{\lfloor (n-1)/2\rfloor, \lceil (n-1)/2\rceil}$. Then add one more vertex and connect it to one vertex of each class. This adds two edges.

4. Suppose that n and d are positive integers. Suppose that $x_1, \cdots, x_n$ are vectors in $\mathbb{R}^d$ with $|x_i| > 1$ for all $1 \le i \le n$. Show that the number of pairs (i,j) with $i < j$ and $|x_i + x_j| < 1$ is at most $\lfloor n^2/4 \rfloor$.

   (Here, $|x|$ is the Euclidean length of the vector $x \in \mathbb{R}^d$; namely

   $$|x| = \sqrt{\sum_{k=1}^{d} x(k)^2},$$

   where $x(k)$ is the kth entry of the vector x.)

   Solution: We want to apply Mantel's theorem. Construct the following graph G. Let $V = \{1, 2, 3, \ldots, n\}$. Add an edge (i,j) if $|x_i + x_j| < 1$. Then if G is triangle-free, by Mantel's theorem, we see that the claim is proved.

   Assume otherwise that G has a triangle and the three vectors are x, y, and z. Hence,

   $$|x+y| < 1, \quad |y+z| < 1, \quad |x+z| < 1.$$

   Moreover,

   $$|x| > 1, \quad |y| > 1, \quad |z| > 1.$$

   Use the above two facts:

   $$\begin{aligned}
   3 &> |x+y|^2 + |y+z|^2 + |x+z|^2 \\
   &= \sum_{k=1}^{n} (x(k)+y(k))^2 + \sum_{k=1}^{n} (y(k)+z(k))^2 + \sum_{k=1}^{n} (z(k)+x(k))^2 \\
   &= \sum_{k=1}^{n} \big(2x(k)^2 + 2y(k)^2 + 2z(k)^2 + 2x(k)y(k) + 2y(k)z(k) + 2x(k)z(k)\big) \\
   &= \sum_{k=1}^{n} x(k)^2 + \sum_{k=1}^{n} y(k)^2 + \sum_{k=1}^{n} z(k)^2 \\
   &\quad + \sum_{k=1}^{n} \big(x(k)^2 + y(k)^2 + z(k)^2 + 2x(k)y(k) + 2y(k)z(k) + 2x(k)z(k)\big) \\
   &> 1 + 1 + 1 + \sum_{k=1}^{n} (x(k)+y(k)+z(k))^2 \\
   &= 3 + \sum_{k=1}^{n} (x(k)+y(k)+z(k))^2 .
   \end{aligned}$$

   This is impossible.