Advanced Combinatorics - 2016 Fall
Solutions to Exercise 2

Comments and corrections are welcome.

Heng Guo
h.guo@qmul.ac.uk

1. Prove that if G is a graph of order $n \geq 3$ such that $d(u)+d(v) \geq n$ for any $(u,v) \notin E(G)$, then G is Hamiltonian.

   Solution: It is pretty much the same proof as the Dirac's theorem. For the connectedness, pick u and v from two different components, then $d(u) + d(v) \leq n-2$. Contradiction.

   The rest of the proof is almost identical. The only thing to note is that x_0 and x_ℓ cannot be adjacent, as otherwise we either have a Hamiltonian cycle or can construct a longer path.

2. Show that Dirac's theorem is the best possible in terms of minimal degrees. In other words, for any odd $n \geq 3$, construct a graph G such that $\delta(G) = \frac{n-1}{2}$ and G is not Hamiltonian.

   Solution: Note that we want to avoid a cycle of length n, which is odd. It is natural to construct a bipartite graph. Then, to spread degrees evenly, consider $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

3. Let $k < n$ be two positive integers. Let G be a connected graph of order n such that $\delta(G) \geq \frac{k}{2}$. Prove that G must have a cycle of length $\left\lceil \frac{k}{2} \right\rceil + 1$, and this is tight.

   Solution: Once again, consider the maximal length path $P_\ell = \{x_0, x_1, \ldots, x_\ell\}$. Consider $\Gamma(x_0)$. By maximality, $\Gamma(x_0) \subset P_\ell \backslash \{x_0\}$. Since $|\Gamma(x_0)| \geq k/2$, there exists a $t \geq \lceil k/2 \rceil$ such that $x_t \in \Gamma(x_0)$ (otherwise there is not enough "room" for $\Gamma(x_0)$). This creates a cycle of length $\lceil k/2 \rceil + 1$.

   This is best possible. Consider a bunch of copies of $K_{\lceil k/2 \rceil + 1}$. This graph satisfies the degree constraint and has no cycles of length $\left\lceil \frac{k}{2} \right\rceil + 2$. However it is not connected. To make it connected without creating cycles, connect these copies by one edge in a tree-like structure.

   In technical terms, we "contract" each $K_{\lceil k/2 \rceil + 1}$ into a vertex. Then construct a spanning tree.