Advanced Combinatorics - 2016 Fall
Solutions to Exercise 3

Comments and corrections are welcome.

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1. Recall the second proof of Turán's Theorem in class. Use Erdős's theorem to show that if a K_{r+1}-free graph G of order n has $t_r(n)$ many edges, then it is isomorphic to $T_r(n)$.

   Solution: Suppose G is a K_{r+1}-free graph with $t_r(n)$ many edges. By Erdős's theorem, there exists an r-partite graph H with $V(H) = V(G)$ and $d_H(v) \geq d_G(v)$ for all $v \in G$. Since $e(G) = t_r(n)$, we must have that $d_H(v) = d_G(v)$.

   Thus the equality must hold in every step. Recall that G_1 is the subgraph induced on $\Gamma(v)$ where v is a vertex such that $d_G(v) = \Delta(G)$. In particular, we have that $d_{G_1}(u) = d_{H_1}(u)$ for all $u \in V(G_1)$.

   Let $U_1 = V(G)\backslash\Gamma(v)$ and $W_1 = \Gamma(v)$. We claim that there is no edges of G within U_1. First notice that

$$\sum_{x \in U_1} d_G(x) = 2e(G[U_1]) + e(U_1, W_1).$$

   This is because each edge within $G[U_1]$ is counted twice in the left hand side, and each edge in $e(U_1, W_1)$ is counted only once. Then

$$\begin{aligned}
e(G) &= e(G[U_1]) + e(U_1, W_1) + e(G_1) \\
&= \sum_{x \in U_1} d_G(x) + e(G_1) - e(G[U_1]) \\
&= \sum_{x \in U_1} d_H(x) + e(H_1) - e(G[U_1]) \\
&= e(H) - e(G[U_1]).
\end{aligned}$$

   The last step is because H is r-partite and U_1 is a class of H. However, we know that $e(G) = e(H)$. Thus $e(G[U_1]) = 0$.

   We can recursively apply the same argument on G_1 to produce U_2 with no edges within. Eventually we stop at some partition $U_1, U_2, \ldots, U_k$. We claim that $k \leq r$. This is because otherwise we get a clique of size $r + 1$, contradicting K_{r+1}-freeness.

2. Recall the third proof of Turán's Theorem in class.
   Let G be a K_{r+1}-free graph that maximizes the number of edges. In class we have shown that if $uw \notin E$, $vw \notin E$, then $uv \notin E$.

Fill in the details to determine G's structure.

Solution: Define an equivalence relation $\sim$ such that $u \sim v$ if $uv \notin E$. It is easy to verify that this is an equivalence relation:

- $v \sim v$.
- $u \sim v$ if and only if $v \sim u$.
- If $u \sim w$, $w \sim v$, then $u \sim v$. (The claim above)

Thus this equivalence relation partitions the whole graph into several equivalence classes. For each class, there is no edge in between. For two different classes, all edges are present. In other words, the graph is a k-partite complete graph for some k.

Then we claim that $k \leq r$. This is because G contains K_k as a subgraph. Thus $k \geq r+1$ would contradict to K_{r+1}-freeness. Hence G is a complete r-partite graph that maximizes the number of edges. It has to be the Turán graph.

3. We know that trees have maximum number of edges in a graph of order n without any cycle, which have $n-1$ edges. Determine:

   (a) The maximum number of edges in a graph of order n without any cycle of odd length.

   (b) The maximum number of edges in a graph of order n without any cycle of even length.

Solution (a): Such a graph G must be triangle-free. Hence by Mantel's theorem $e(G) \leq \lfloor n^2/4 \rfloor$. Moreover, $e(G) = \lfloor n^2/4 \rfloor$ is achieved by complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, and a bipartite graph does not contain any odd cycle. Hence the answer is exactly $\lfloor n^2/4 \rfloor$.

Solution (b): We claim that if G has no even cycle, then $e(G) \leq \lfloor 3(n-1)/2 \rfloor$. We prove the claim by an induction on n. The base case of $n=2$ is trivial.

Now for the induction step. Let $n \geq 3$, and the claim holds for graphs of $\leq n-1$ vertices. If G has no cycle, then G is a tree and $e(G) \leq n-1 < 3(n-1)/2$.

Hence we may assume that G has a cycle of odd length. Let C be that cycle. Let G' be the graph obtained by removing all edges in C (but not the vertices!). If D is a component of G', then G' can contain at most one vertex of C. This is because otherwise it contains say v_1 and v_2 on C. Because D is connected, so there is a path from v_1 to v_2. No matter what the parity of this path is, there is a path in C has the same parity. Thus we can construct a cycle of even length.

If C is a component of G. Then removing C and applying the induction hypothesis yields the claim.

Otherwise, let A be a component of G' of more than one vertex and sharing one vertex with C. Let B be the rest of the graph except this special vertex on C. Let $|A| = a$ and $|B| = b$. Then $a + b - 1 = n$. Apply the induction hypothesis on A and B:

$$\begin{aligned} e(G) &= e(A) + e(B) \\ &\leq \lceil 3(a-1)/2 \rceil + \lceil 3(b-1)/2 \rceil \\ &\leq \lceil 3(a+b-2)/2 \rceil \\ &= \lceil 3(n-1)/2 \rceil . \end{aligned}$$

Thus the claim holds.

We still need to construct graphs without even cycles achieving this bound. The idea is to take triangles and then "stick" them together, sharing only one vertex. This is pictured in Figure 1a. If we have k triangles, then this construction have $3k$ many edges and $3k - (k-1) = 2k + 1$ many vertices. This finishes the odd n case.

For even n, we take the same construction, but attach the special vertex with one more new vertex. There are $3k + 1$ many edges and $2k + 2$ vertices. This is pictured in Figure 1b. The bound is also met.

(a) Odd n (b) Even n

Figure 1: Sticking triangles

4. Let A be the graph in Figure 2a and B be the graph in Figure 2b.

(a) A (b) B

Figure 2: The graphs A and B

(a) If G is a graph on $n \geq 4$ vertices with $e(G) \geq \lfloor n^2/4 \rfloor + 1$, then G contains A as a subgraph.

(b) If G is a graph on $n \geq 5$ vertices with $e(G) \geq \lfloor n^2/4 \rfloor + 2$, then G contains B as a subgraph.

(c) Determine $ex(n, A)$ and $ex(n, B)$.

Solution (a): Let G be a graph of order $n \geq 4$ and A-free. We show that $e(G) \leq \lfloor n^2/4 \rfloor$. By Mantel's theorem, the inequality holds if G is triangle-free. Thus we only care about G that contains at least one triangle.

We do an induction on n. The base case is that $n = 4$. As G contains a triangle, say uvw. Then the other vertex can only be adjacent to one of u, v, w, as otherwise it produces A. Hence $e(G) \leq 4$ as desired.

Let $n \geq 5$ and assume that the claim holds with $\leq n - 1$ vertices. Let uvw be the triangle of G. We claim that

$$d(u) - 2 + d(v) - 2 + d(w) - 2 \leq n - 3.$$

This is because the three sets $\Gamma(u)\backslash\{v, w\}$, $\Gamma(v)\backslash\{u, w\}$, and $\Gamma(w)\backslash\{u, v\}$ have to be mutually disjoint.

Thus, say $d(v)$ is the smallest of these three. Then $d(v) \leq (n + 1)/3 \leq \lfloor n/2 \rfloor$ as $n \geq 5$. We can simply remove v and proceed as in Mantel's theorem.

Solution (b): The idea is the same as the proof above. Let G be a graph of order $n \geq 5$ and B-free. We show that $e(G) \leq \lfloor n^2/4 \rfloor + 1$. By part (a), the inequality holds if G is A-free. Thus we only care about G that contains at least one A. Then we do an induction on n.

The base case is $n = 5$. Suppose the vertices are $uvwxy$ and uvw, uvx are the two triangles. If $wx \in E$, then y can be adjacent to at most one of $uvwx$, implying $e(G) \leq 6 + 1$ as desired. Otherwise $wx \notin E$. If $d(y) \geq 3$, then it is adjacent to, say, u and w. Then we have a copy of B. Hence $d(y) \leq 2$ and $e(G) \leq 5 + 2 \leq 7$ as desired.

For the induction step. Again we only need to find a vertex of degree at most $\lfloor n/2 \rfloor$. Since G contains a copy of A, let uvw be a triangle uvx be another. If $d(w) \leq 3$, then we can remove it as its degree is no more than $\lfloor n/2 \rfloor$. Otherwise $d(w) > 3$, then there is another $y \in V(G)$ such that $wy \in E$. Now notice that w and y cannot have any common neighbour as otherwise we have a B. Then we have that $d(w) - 1 + d(y) - 1 \leq n - 2$. Hence $d(w) + d(y) \leq n$. The smaller will be at most $\lfloor n/2 \rfloor$ and we then proceed as in Mantel's theorem.

Solution (c): Part (a) showed that $ex(n, A) \leq \lfloor n^2/4 \rfloor$. Since the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ achieves this bound and does not contain A, we have that $ex(n, A) = \lfloor n^2/4 \rfloor$.

Part (b) showed that $ex(n, A) \leq \lfloor n^2/4 \rfloor + 1$. To get a matching bound, we take the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ with an arbitrary single edge. It will create a lot of triangles, but not the graph B. Thus we have that $ex(n, A) = \lfloor n^2/4 \rfloor + 1$.