Advanced Combinatorics - 2016 Fall
Solutions to Exercise 4

Comments and corrections are welcome.

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1. Let $r, n \in \mathbb{N}$ with $r \geq 2$. Let G be a graph with n vertices and with $\delta(G) \geq \lfloor (1-\frac{1}{r})n \rfloor + 1$. Show that G contains a K_{r+1}.

   Solution: We will use Turán's theorem. Let d be the minimum degree of $T_r(n)$; that is, $d = n - \lceil n/r \rceil$. Thus $d = \lfloor n - n/r \rfloor = \lfloor (1 - 1/r)\, n \rfloor$, and $\delta(G) \geq d + 1$.

   In $T_r(n)$, all vertices have degrees either d or $d+1$, and at least one vertex has degree d. (The latter case is when r divides n.) Thus by the Handshaking lemma, we see that

   $$e(G) = \frac{1}{2} \sum_{v \in V(G)} d_G(v) > \frac{1}{2} \sum_{v \in V(T_r(n))} d_{T_r(n)}(v) = t_r(n).$$

   In other words, G has more edges than $T_r(n)$. Hence by Turán's theorem, G contains a copy of K_{r+1}.

2. Let $n \geq r \geq 2$ be two positive integers. Prove that

   $$\left(1 - \frac{1}{r}\right) \frac{n^2}{2} \geq t_r(n) \geq \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

   Solution: Recall the definition of Turán's graph. Let n, r be two integers such that $n \geq r \geq 2$. Let $n = kr + s$ where $k \geq 1$ and $0 \leq s < r$. Then the Turán graph $T_r(n)$ is the complete r-partite graph where s many classes have size $k+1$ and $r-s$ many classes have size k.

   How many edges are there in $T_r(n)$? We can calculate as follows:

   $$t_r(n) = \binom{n}{2} - (r-s)\binom{k}{2} - s\binom{k+1}{2},$$

   by considering the edges that are missing. Thus we have that

   $$\begin{aligned} \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - t_r(n) &= \left(1 - \frac{1}{r}\right) \frac{n^2}{2} - \binom{n}{2} + (r-s)\binom{k}{2} + s\binom{k+1}{2} \\ &= -\frac{1}{r} \cdot \frac{n^2}{2} + \frac{n}{2} + (r-s)\binom{k}{2} + s\binom{k+1}{2}. \end{aligned}$$

Recall that $n = kr + s$. Plug it in:

$$\begin{aligned}
\left(1-\frac{1}{r}\right)\frac{n^2}{2} - t_r(n) &= -\frac{1}{r}\cdot\frac{(kr+s)^2}{2} + \frac{kr+s}{2} + (r-s)\frac{k(k-1)}{2} + s\frac{k(k+1)}{2} \\
&= -\frac{k^2r + 2ks + s^2/r}{2} + \frac{kr+s}{2} + \frac{(r-s)k^2 - (r-s)k + sk^2 + sk}{2} \\
&= -\frac{k^2r + 2ks + s^2/r}{2} + \frac{kr+s}{2} + \frac{rk^2 - (r-2s)k}{2} \\
&= \frac{-k^2r - 2ks - s^2/r + kr + s + rk^2 - rk + 2sk}{2} \\
&= \frac{s - s^2/r}{2} = \frac{s}{r}\cdot\frac{r-s}{2} \geq 0
\end{aligned}$$

For the lower bound, we use the result above and have that

$$\begin{aligned}
\left(1-\frac{1}{r}\right)\binom{n}{2} - t_r(n) &= \left(1-\frac{1}{r}\right)\frac{n^2}{2} - t_r(n) - \left(1-\frac{1}{r}\right)\frac{n}{2} \\
&= \frac{s - s^2/r}{2} - \left(1-\frac{1}{r}\right)\frac{kr+s}{2} \\
&= \frac{s - s^2/r - s + s/r}{2} - \left(1-\frac{1}{r}\right)\frac{kr}{2} \\
&= -\frac{s(s-1)}{2r} - \left(1-\frac{1}{r}\right)\frac{kr}{2} \leq 0.
\end{aligned}$$

3. Let P be the Petersen graph, as shown in Figure 1. Show that $\pi(P) = 1/2$.

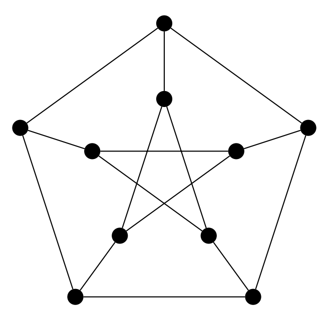

Figure 1: The Petersen graph

Some background: Some optimistic people conjectured that every bridgeless cubic graph is three-edge-colourable. However, the Petersen graph is the smallest counter example.

Solution: Due to the Erdős-Stone theorem, all we need to show is that $\pi(P) = 3$. Clearly P is not bipartite since it contains cycles of length 5. A 3-colouring is shown in Figure 2.

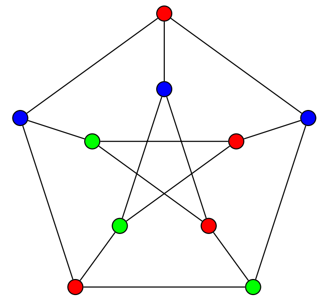

Figure 2: The Petersen graph

4. Let G be a graph with average degree D. That is,

$$\frac{1}{|V(G)|}\sum_{v\in V(G)} d(v) = D.$$

Show that G has a graph of minimum degree at least $D/2$.

Solution: Suppose the claim does not hold. That is, every subgraph of G has minimum degree less than $D/2$. Then we can construct a sequence of graphs

$$G_n, G_{n-1}, \ldots, G_1$$

as follows. Let $G_n = G$. Given G_i, construct G_{i-1} by choosing some vertex $v_i \in V(G_i)$ with minimum degree in G_i, and setting $G_{i-1} = G_i - v_i$. By our assumption, G_i has minimum degree less than $D/2$, so v_i has degree less than $D/2$ in G_i. It implies that $e(G_i) - e(G_{i-1}) < D/2$ for all $2 \le i \le n$. Since G_1 has just one vertex, we must have that $e(G_1) = 0$, and

$$\begin{aligned} e(G_n) &= e(G_1) + \sum_{i=2}^{n}(e(G_i) - e(G_{i-1})) \\ &< 0 + \sum_{i=2}^{n}\frac{D}{2} = \frac{(n-1)D}{2} < \frac{nD}{2}. \end{aligned}$$

However, we know that $e(G) = \frac{nD}{2}$. This is a contradiction.

5. We know that $\pi(A) = 1 - \frac{1}{r}$ if $\chi(A) = r+1$. Thus $\lim_{n\to\infty}\frac{ex(n,K_{r+1})}{ex(n,A)} = 1$.

Q: For each integer $r > 1$, find a graph A with $\pi(A) = 1 - \frac{1}{r}$, but $ex(n, A) > ex(n, K_{r+1})$.

Solution: Let $A = T_{r+1}(2(r+1)) = K^2_{r+1}$, the $(r+1)$-partite Turán graph with 2 vertices in each class. Clearly $\pi(A) = 1 - 1/r$. Let G be the graph obtained by taking $T_r(n)$ and adding just one edge connecting two vertices in one of the larger classes. Then $e(G) = t_r(n) + 1$, but we claim that G is A-free. It implies that $ex(n, A) \ge e(G) > t_r(n)$.

To see the claim, assume that we have a copy of A in G. Since A has $2r+2$ vertices and $T_r(n)$ has only r classes, there are at least three vertices of this copy of A in the same class of $T_r(n)$. Say these three vertices are u, v, and w. By our construction, we added only one edge inside any class of $T_r(n)$. Hence one of the three vertices, say u, is not adjacent to v and w. This is impossible in A. Any vertex in A is adjacent to all but one vertex.

6. We have seen that $\pi(K_{r+1}) = 1 - \frac{1}{r}$.

   On the other hand, it is not necessary to contain K_{r+1} to have density $1 - \frac{1}{r}$.

   Q: For each integer $r > 1$, find a graph B which is K_{r+1}-free but has $\pi(B) = 1 - \frac{1}{r}$.

   Solution: We construction this graph by induction on r. For $r = 2$, consider a cycle of length 5. It is not bipartite and in fact has chromatic number 3. In the mean time it does not contain K_3, a triangle.

   For the induction step, suppose that we have a graph B_{r-1} with $\chi(B_{r-1}) = r$ and B_{r-1} is K_r-free. Construct B_r by adding a new vertex v that is adjacent to all vertices of B_{r-1}. Then B_r is K_{r+1}-free and has chromatic number $r+1$, as required.