Advanced Combinatorics - 2016 Fall
Solutions to Exercise 5

Comments and corrections are welcome.

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1. Show that a C_4-free graph with 13 vertices has at most 26 edges.

   Solution: Apply the theorem we showed in the class. We get that

$$ex(13, C_4) \le \frac{13}{4}\left(\sqrt{4 \times 13 - 3} + 1\right) = 26.$$

2. Let G be a graph with n vertices. Show that we can cover all edges of G with a collection of a triangles of G and b edges of G, for some non-negative integers a and b such that $a + b \le \lfloor n^2/4 \rfloor$.

   Solution: We prove the claim by an induction on n. The base case is when $n = 2$, we can easily cover the only possible edge. If $n = 3$, then $\lfloor n^2/4 \rfloor = 2$. If there are three edges in the graph, it must be triangle. Thus we can cover the graph by either one triangle or two edges.

   For the induction step, there are at least one edge. Let uv be an edge. The number of triangles in G containing uv is exactly $T := |\Gamma(u) \cap \Gamma(v)|$. Notice that

$$\begin{aligned} n &\ge |\Gamma(u) \cup \Gamma(v)| \\ &= |\Gamma(u)| + |\Gamma(v)| - |\Gamma(u) \cap \Gamma(v)| \\ &= d(u) + d(v) - T. \end{aligned} \tag{1}$$

   Let $G' = G - \{u, v\}$. By the induction hypothesis, there are a' triangles and b' edges such that G' is covered and $a' + b' \le \lfloor (n-2)^2/4 \rfloor = \lfloor n^2/4 \rfloor - (n-1)$. Thus we are allowed to use $n - 1$ many extra triangles and edges in total.

   We cover all triangles containing uv by T triangles. Then we can cover all other edges adjacent to u or v edge by edge. If $T = 0$, then there is no triangle and we still need to cover uv. This case is the same as Mantel's theorem. By (1), $d(u) + d(v) \le n$ and the extra number of edges is

$$(d(u) - 1) + (d(v) - 1) + 1 = d(u) + d(v) - 1 \le n - 1,$$

   as desired.

   Otherwise $T \ge 1$ and we do not need to cover uv by using single edges. The extra number of edges and triangles we need is

$$T + (d(u) - T - 1) + (d(v) - T - 1) = d(u) + d(v) - T - 2 \le n - 2,$$

   where we also use (1).

3. Let H be a fixed bipartite graph and $n \in \mathbb{N}$. Let $bi\text{-}ex(n, H)$ be the maximum number of edges in an H-free bipartite graph with vertex classes L and R such that $|L| = |R| = n$. Show that

$$bi\text{-}ex(n, C_4) \leq \frac{1}{2} n(\sqrt{4n-3} + 1).$$

Moreover, construct an infinite family of graphs such that the equality holds.

Solution: Let G be a C_4-free bipartite graph with vertex classes R and L such that $|R| = |L| = n$. Let $m = e(G)$. We count the number N of paths of length 2 in two different ways. As in the lecture, we have that

$$N = \sum_{v \in V} \binom{d(v)}{2},$$

as each vertex v can be the middle point of $\binom{d(v)}{2}$ paths of length 2.

On the other hand, there is at most one such path for any two vertices u and v. However, if we choose u and v from R and L respectively, there is no such path at all. Thus either u, v are both from R or they are both from L. In each case there are $\binom{n}{2}$ many choices. It implies that

$$N \leq 2\binom{n}{2}.$$

Putting everything together, we have that

$$\sum_{v \in V} \binom{d(v)}{2} = N \leq 2\binom{n}{2}.$$

As in the lecture, the left hand side can be lower bounded as follows,

$$\begin{aligned} \sum_{v \in V} \binom{d(v)}{2} &= \frac{1}{2} \sum_{v \in V} d(v)^2 - \frac{2m}{2} \\ &\geq \frac{\left(\sum_{v \in V} d(v)\right)^2}{4n} - m \\ &= \frac{m^2}{n} - m. \end{aligned}$$

Thus we have that

$$m^2 - nm - n^2(n-1) \leq 0,$$

implying that

$$m \leq \frac{n}{2} \left(\sqrt{4n-3} + 1\right).$$

This proves the claim.

To construct an infinite family of graphs with a holding equality, note that the example we constructed in lectures is indeed bipartite. In fact, we can verify that the equality does hold.

Let P and L be the points and lines in the Desarguesian projective plane of order p where p is a prime, which we have constructed in lecture. Let $n = p^2+p+1 = |P| = |L|$. The corresponding graph has $m = (p+1)(p^2+p+1) = (p+1)n$ edges. Solving the quadratic equation

$$p^2 + p - (n-1) = 0,$$

we get

$$p = \frac{\sqrt{4n-3}-1}{2}.$$

Thus

$$m = (p+1)n = \frac{n}{2}\left(\sqrt{4n-3}+1\right),$$

as required.

4. (a) Let S be a subset of $\{1, 2, \ldots, n\}$ with the property that no integer (not necessarily in S) can be written as the sum of two elements of S in more than one way. (It is okay that some integer is not the sum of any two elements of S.) Use the fact that

$$ex(n, C_4) \leq \frac{n}{4}(\sqrt{4n-3}+1)$$

to prove that

$$|S| \leq c_0\sqrt{n},$$

for some positive constant c_0.

(b) Construct such an S whose order of magnitude is as large as you can.

Solution: Let $S \subseteq \{1, 2, \ldots, n\}$ be the set with the desired property. In fact, these sets are named *Sidon* sets, after Hungarian mathematician Simon Sidon. Let $G = (V, E)$ be a bipartite graph constructed as follows. Let $V = R \cup L$ where R and L are disjoint. Let

$$R := \{a_i \mid i \in \mathbb{Z},\ -(n-1) \leq i \leq n\};$$
$$L := \{b_i \mid i \in \mathbb{Z},\ -(n-1) \leq i \leq n\}.$$

Then $|R| = |L| = 2n$. Let $a_i b_j$ be an edge if $i + j \in S$.

We claim that G is C_4-free. Suppose otherwise and $a_i - b_j - a_k - b_\ell - a_i$ is a C_4. Then we have that

$$\begin{aligned} &i + j \in S; && j + k \in S; \\ &k + \ell \in S; && \ell + i \in S. \end{aligned}$$

However, notice that

$$(i + j) + (k + \ell) = (j + k) + (\ell + i).$$

This is in contradiction to no integer can be written as a sum of two integers from S in two ways.

Therefore, we can apply the theorem from lecture, or indeed the question 3 above. Note that each class of G has $2n$ vertices. We have that

$$e(G) \le n(\sqrt{8n - 3} + 1).$$

On the other hand, we claim that $e(G) \ge n\,|S|$. Note that each edge is associated with exactly one element in S, but for each element $s \in S$, there are $2n - s + 1$ edges in G associated with s. This is because

$$s = n + (s - n) = n - 1 + (s - n + 1) = \cdots = 0 + s = \cdots = (s - n) + n.$$

Thus, $a_n b_{s-n}, \ldots, a_0 b_s, \ldots, a_{s-n} b_n$ are all in $E(G)$ and these edges are distinct. As $s \le n$, $2n - s + 1 > n$. Thus $e(G) \ge n\,|S|$.

Putting everything together, we get that

$$n\,|S| \le e(G) \le n(\sqrt{8n - 3} + 1),$$

implying that

$$|S| \le e(G)/n \le \sqrt{8n - 3} + 1 \le 2\sqrt{2n}.$$

To construct a Sidon set S, we may simply take $1, 2, 4, 8, \ldots, 2^{\lfloor \log_2 n \rfloor}$. The claimed property holds, as otherwise there are i, j, k, ℓ such that

$$2^i + 2^j = 2^k + 2^\ell.$$

We may assume that $\ell > i$ and $\ell > j$. Then

$$2^\ell + 2^k \ge 2^{\ell - 1} + 2^{\ell - 1} + 1 \ge 2^i + 2^j + 1.$$

Contradiction! Thus we have a lower bound that $\max |S| \ge \lfloor \log_2 n \rfloor$. It is non-trivial but still far away from the upper bound of $O(\sqrt{n})$.

Let us try to build some dense Sidon set. From this point on it is optional material. The following construction is due to Erdős. We first claim that $\Gamma = \{(x, x^2) \mid x \in \mathbb{Z}\}$ is a Sidon set. In fact, if

$$(x, x^2) + (y, y^2) = (w, w^2) + (z, z^2),$$

for $x < y$ and $w < z$, then $x + y = w + z$ and $x^2 + y^2 = w^2 + z^2$, or equivalently $x^2 - w^2 = z^2 - y^2$. If $x - w = z - y \neq 0$, then dividing $x^2 - w^2 = z^2 - y^2$ by $x - w = z - y$, we have that $x + w = z + y$ which is impossible. Thus $x = w$ and $y = z$.

We still need to map Γ to a subset of $\{1, \ldots, n\}$. The first idea is to map (x, x^2) to a number $1 + x + qx^2$ where q is an integer that is larger than the sum of any two x and y. Say we have x from $0, 1, 2, \ldots, p - 1$. Then $q = 2p$ will do. Thus if

$$(1 + x + 2px^2) + (1 + y + 2py^2) = (1 + w + 2pw^2) + (1 + z + 2pz^2),$$

we have that

$$x + y - w - z = 2p(w^2 + z^2 - x^2 - y^2).$$

Thus if $|w^2 + z^2 - x^2 - y^2| > 1$, then $|x + y - w - z| > 2p$, which is impossible.

Clearly we get a sequence of length p this way. On the other hand, the largest element is $1 + p - 1 + 2p(p - 1)^2 = O(p^3)$. Thus $n = O(p^3)$, or in other words, the length of the Sidon sequence is $\Omega(n^{1/3})$.

To improve it to $\Omega(n^{1/2})$, the trick is that all of the above still holds in a finite field. Let p be an odd prime number. Let $[x]$ be the x modulo p. That is, $[x] \equiv x \bmod p$, and $0 \leq [x] \leq p - 1$. Let

$$S := \{1 + x + 2p[x^2] \mid 0 \leq x \leq p - 1\} \subseteq \{1, \ldots, 2p^2\}.$$

Then $|S| = p$ which is of the order of $\sqrt{n}$ as $n = 2p^2$ here. If S is not a Sidon set, then we have some $x < y$ and $w < z$ such that $x \neq w$ and

$$1 + x + 2p[x^2] + 1 + y + 2p[y^2] = 1 + w + 2p[w^2] + 1 + z + 2p[z^2],$$

which is equivalent to

$$(x + y) - (w + z) = 2p(([w^2] + [z^2]) - ([x^2] + [y^2])).$$

The same reasoning applies. If the right hand side is not 0, then it is at least $2p$ or at most $-2p$. However, the left hand side cannot exceed $2p - 2$ nor be below $-2p + 2$. Thus we have that $x + y = w + z \mod p$ and $x^2 + y^2 \equiv w^2 + z^2 \mod p$. It implies that in the finite field $\mathbb{Z}_p$ we have the following system,

$$\begin{aligned} x + y &= w + z \\ x^2 + y^2 &= w^2 + z^2. \end{aligned}$$

By exactly the same argument as before (the way of solving this system above works in any field), this is impossible unless $x = w$ and $y = z$.

In fact, let $s(n) = \max|S|$ for Sidon set S. It is known that $\lim_{n\to\infty} \frac{s(n)}{\sqrt{n}} = 1$, so $\sqrt{n}$ is asymptotic to $s(n)$. The upper bound is given by Erdős and Turán in 1941 and the lower bound by Ruzsa in 1993.

In the mean time, we may also consider a random subsequence. Say we pick every element with probability p. There are $O(n^3)$ quadruples of (x, y, w, z) such that $x+y = w + z$. Thus the expected number of these "violations" is $O(n^3p^4)$. Also note that the expected length of a random subsequence is pn. We are going to remove one element from each "violating quadruple", which leaves a Sidon sequence of expected length $pn - O(n^3p^4)$. Suppose $p = n^{-\varepsilon}$. Then optimizing the exponent $(1 - \varepsilon = 3 - 4\varepsilon)$ gives $\varepsilon = 2/3$ and the length is $n^{1/3}$.