Advanced Combinatorics - 2016 Fall
Solutions to Exercise 6

Comments and corrections are welcome.

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1. If there is a real $0 \le p \le 1$ such that
$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$
then the Ramsey number $R(k,\ell)$ satisfies that $R(k,\ell) > n$.

   Using this, show that
$$R(4,\ell) \ge \Omega(\ell^{3/2}(\log \ell)^{-3/2}).$$

   Solution: Follow the procedure of the basic method. We colour every edge in a complete graph K_n red with probability p and blue with probability $1-p$. Let R_S be the "bad" event that $S \subset [n]$ is a red clique, and B_S be the "bad" event that $S \subset [n]$ is a blue clique. We have that $\Pr(R_S) = p^{\binom{k}{2}}$ if $|S| = k$, and $\Pr(B_S) = (1-p)^{\binom{\ell}{2}}$ if $|S| = \ell$. Then by the union bound,
$$\begin{aligned}\Pr\left(\bigwedge_{S,\,|S|=k} \overline{R_S} \wedge \bigwedge_{S,\,|S|=\ell} \overline{B_S}\right) &= 1 - \Pr\left(\bigvee_{S,\,|S|=k} R_S \vee \bigvee_{S,\,|S|=\ell} B_S\right) \\ &\ge 1 - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} > 0.\end{aligned}$$
Thus there must exist an edge colouring of the complete graph so that there is no red clique of size k nor blue clique of size ℓ.

   For the case of $k = 4$, we need to choose p appropriately so that there exists a c such that $n = c \cdot \left(\frac{\ell}{\log \ell}\right)^{3/2}$ and
$$\binom{n}{4} p^{\binom{4}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1.$$
Note that
$$\begin{aligned}\binom{n}{4} p^{\binom{4}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} &\le \left(\frac{ne}{4}\right)^4 p^6 + \left(\frac{ne}{\ell}\right)^\ell e^{-p\binom{\ell}{2}} \\ &= \left(\frac{ce}{4} \cdot \left(\frac{\ell}{\log \ell}\right)^{3/2}\right)^4 p^6 + \left(\frac{ce}{\ell} \cdot \left(\frac{\ell}{\log \ell}\right)^{3/2}\right)^\ell e^{-p\binom{\ell}{2}} \\ &= \left(\frac{ce}{4}\right)^4 \left(\frac{\ell}{\log \ell} \cdot p\right)^6 + \left(\frac{ce}{\ell} \cdot \left(\frac{\ell}{\log \ell}\right)^{3/2}\right)^\ell e^{-p\binom{\ell}{2}}.\end{aligned}$$

To make the first term a constant, we can choose $c = e^{-1}$ and $p = \frac{\log \ell}{\ell}$, in which case the first term becomes 4^{-4}. Plugging it back, we see that the second term becomes

$$\begin{aligned}\left(\frac{ce}{\ell}\cdot\left(\frac{\ell}{\log \ell}\right)^{3/2}\right)^{\ell} e^{-p\binom{\ell}{2}} &= \left(\frac{1}{\ell}\cdot\left(\frac{\ell}{\log \ell}\right)^{3/2}\right)^{\ell} e^{-\frac{\log \ell}{\ell}\cdot\frac{\ell(\ell-1)}{2}} \\ &= \ell^{\ell/2}\left(\log \ell\right)^{-3/2\ell} e^{-\log \ell\cdot\frac{\ell-1}{2}} \\ &= \ell^{\ell/2}\left(\log \ell\right)^{-3/2\ell} \ell^{-\frac{\ell-1}{2}} \\ &= \ell^{1/2}\left(\log \ell\right)^{-3/2\ell}.\end{aligned}$$

Thus with our choices of c and p, we have that

$$\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} \leq 4^{-4} + \ell^{1/2}\left(\log \ell\right)^{-3/2\ell} < 1,$$

where the last equality is easy to verify for all $\ell \geq 4$.

2. Let $k \geq 4$ and H be a k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there is a 4-colouring of the vertices of H so that in every edge all four colours are represented.

   Remark: this kind of colourings is called "rainbow" colouring.

   Solution: We colour every vertex with one of the four colours uniformly at random. Let A_e be the event that e does not contain all four colours, where e is a hyperedge. When A_e holds, e must be coloured by at most 3 colours, and there are $\binom{4}{3}$ many ways of picking the 3 colours. Thus, by a union bound, for any hyperedge e,

$$\Pr(A_e) \leq \frac{\binom{4}{3}\cdot 3^k}{4^k} = \frac{3^k}{4^{k-1}}.$$

   Then by the union bound,

$$\begin{aligned}\Pr\left(\bigwedge_{e\in E}\overline{A_e}\right) &= 1 - \Pr\left(\bigvee_{e\in E}\overline{A_e}\right) \\ &\geq 1 - m\cdot\frac{3^k}{4^{k-1}} > 0,\end{aligned}$$

   where we use the assumption that $m < \frac{4^{k-1}}{3^k}$. Thus there must exist at least one valid rainbow colouring of H.

3. Let S be a finite collection of binary strings. For example, S may contain elements like 000, 010101, etc. Assume that no member of S is a prefix of another. Let N_i denote the number of strings of length i in S. Show that

$$\sum_{i=1}^{\infty}\frac{N_i}{2^i} \leq 1.$$

Solution: Suppose the maximum length of elements in S is ℓ. Define the following probability process, starting with $s = \emptyset$:

(a) Uniformly at random generate one bit $x = 0/1$ and append x to s.

(b) If the current s is an element in S, then stop and return success. If s has length ℓ but $s \notin S$, then stop and return failure. In any other case, go back to (a).

We claim that the probability that such a trial stops at length $i \leq \ell$ successfully is $p_i := \frac{N_i}{2^i}$. This is because that no element of S is a prefix of any other. Let $s_1, \ldots, s_{N_i}$ be the strings in S with length i. Then the probability of stopping with s_k for any $1 \leq k \leq N_i$ is 2^{-i}, and these events are disjoint.

$$p_i = \sum_{k=1}^{N_i} \frac{1}{2^i} = \frac{N_i}{2^i}.$$

Clearly the probability of success is at most 1. It implies that

$$\sum_{i=1}^{\ell} \frac{N_i}{2^i} = \sum_{i=1}^{\ell} p_i \leq 1.$$

4. A *planar graph* is one so that we can draw it on a plane. By Euler's formula, for a planar graph G, $m \leq 3n - 6$, where n is the number of vertices and m is the number of edges. Moreover, if a graph is drawn on a plane, then there are at least $m - 3n + 6 > m - 3n$ many crossings.

   Use the above fact, show that if $m \geq 4n$, then drawing G on a plane has at least $\frac{m^3}{64n^2}$ many crossings.

   Solution: Let t be the number of crossings for an arbitrary drawing of G on a plane. Consider a random subgraph G' which is the induced subgraph of choosing every vertex with probability p. Let X be the number of vertices, Y be the number of edges, and Z be the number of crossings in G'. (All of them are random variables.) Then $\mathbb{E}\, X = pn$, $\mathbb{E}\, Y = p^2 m$, and $\mathbb{E}\, Z = p^4 t$. By the linearity of expectation,

$$\mathbb{E}[Z - (Y - 3X)] = p^4 t - (p^2 m - 3pn).$$

   There must exist a G' for which $Z - (Y - 3X)$ is at least $p^4 t - (p^2 m - 3pn)$. Clearly G' is still drawn on a plane. Thus, using the fact, we have that $Z > Y - 3X$. In other words,

$$p^4 t > p^2 m - 3pn.$$

   Equivalently, $t > p^{-2} m - 3p^{-3} n$. We want $p^{-2} m$ and $p^{-3} n$ to have the same order, yielding $p = C \cdot \frac{n}{m}$ for some constant C. Setting $C = 4$ proves the original claim.

   This fact is called the "crossing lemma", and is found by Ajtai, Chvátal, Newborn, and Szemerédi in 1982, and independently by Leighton in 1983.