Advanced Combinatorics - 2016 Fall
Solutions to Exercise 7

Comments and corrections are welcome.

Heng Guo
h.guo@qmul.ac.uk

1. Let $k \geq 2$ be an integer. Let $H = (V, E)$ be a k-uniform hypergraph (every hyperedge has size exactly k) with $|E| = 4^{k-1}$. Show that there is a colouring of V by four colours so that no edge is monochromatic.

   Solution: We colour each vertex using 4 colours independently and uniformly at random. For every edge e, let X_e be indicator variable of the event that e is monochromatic. Thus,

   $$\mathbb{E}\, X_e = \Pr(X_e = 1) = 4 \cdot 4^{-k}.$$

   Let X be the total number of monochromatic edges; namely $X = \sum_{e \in E} X_e$. By the linearity of expectations,

   $$\mathbb{E}\, X = \sum_{e \in E} \mathbb{E}\, X_e = |E| \cdot 4^{1-k} = 1.$$

   We claim that it cannot be that $X = 1$ for all random choices. This is obvious, as we can colour all vertices by the same colour, making $X = |E|$. As a consequence, there must be some colouring such that $X < 1$. However, X is an integer and it must be that $X = 0$. Therefore there exists a colouring such that no edge is monochromatic.

   Remark: This shows a subtle difference between the basic method and using expectations. Applying the basic method to this problem, the probability of what we want (using an union bound) is exactly 0.

2. Let $G = (V, E)$ be a bipartite graph where $V = L \cup R$, and L and R are the two classes. Suppose that $d(v) \geq 1$ for every $v \in L$.

   Prove that there exists a subset S of vertices such that $|S| \geq n/2$ and $d_{G'}(v)$ is odd for every $v \in L \cap S$, where G' is the induced subgraph on S.

   Solution: We will put vertices from R into S independently and uniformly at random. Then, we will put every vertex $v \in L$ into S if v has an odd number of neighbours in $R \cap S$. Thus, this set S will satisfy the condition that $d_{G'}(v)$ is odd for every $v \in L \cap S$. Let X_v be the indicator variable of the event that $v \in S$. For every $v \in R$,

   $$\mathbb{E}\, X_v = \frac{1}{2}.$$

For every $u \in L$, it is chosen if an odd number of its neighbours are chosen. Since $d(v) \geq 1$, it happens with probability exactly $1/2$. One way of seeing it is the fact that

$$\binom{d}{0} - \binom{d}{1} + \binom{d}{2} - \cdots + (-1)^d \binom{d}{d} = (1-1)^d = 0,$$

where $d = d(v)$, which implies that

$$\sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{2i} = \sum_{i=0}^{\lceil d/2 \rceil - 1} \binom{d}{2i+1}.$$

Thus, for $u \in L$, we also have that

$$\mathbb{E}\, X_u = \frac{1}{2}.$$

Hence, by the linearity of expectation,

$$\mathbb{E}\,|S| = \sum_{v \in V} \mathbb{E}\, X_v = |V|/2 = n/2.$$

There must exist an S such that $|S| \geq n/2$.

3. Let $G = (V, E)$ be a bipartite graph with n vertices. Each vertex $v \in V$ is associated with a list of colours, denoted $S(v)$ and $|S(v)| \geq \log_2 n$.

   Show that there is a proper colouring of G by assigning each vertex a colour from $S(v)$.

   Solution: The problem about list-colouring is that these lists may conflict each other. We need to find a way to colour the graph properly without using anything special about the lists.

   Crucially, we will use the fact that the graph is bipartite. Let $V = L \cup R$ be the bipartition. Let $S = \bigcup_{v \in V} S(v)$ be the total set of colours. For every colour $c \in S$, assign it to two new sets S_L or S_R uniformly at random. This creates a random bipartition of S into S_L and S_R. Then if we colour $v \in L$ using colours in $S(v) \cap S_L$ and $u \in R$ using colours in $S(u) \cap S_R$, the colouring will be proper.

   In other words, we can find a proper colouring if $S(v) \cap S_L \neq \emptyset$ for every $v \in L$ and $S(u) \cap S_R \neq \emptyset$ for every $u \in R$. Let X_v be indicator variable of the event that $S(v) \cap S_L = \emptyset$ if $v \in L$ or $S(v) \cap S_R = \emptyset$ if $v \in R$. For this to happen, all colours in $S(v)$ must be classified into the wrong set. Since $|S(v)| \geq \log_2 n$,

$$\mathbb{E}\, X_v = \Pr(X_v = 1) = 2^{-|S(v)|} \leq \frac{1}{n}.$$

Let X be the total number of vertices such that they do not have a colour to choose from; that is $X = \sum_{v \in V} X_v$. By the linearity of expectations,

$$\mathbb{E}\, X = \sum_{v \in V} \mathbb{E}\, X_v \leq n \cdot \frac{1}{n} = 1.$$

Moreover, it is impossible that $X = 1$ for all random choices of S_L and S_R. Thus, there exists a bipartition S_L and S_R such that $X < 1$, implying $X = 0$. As explained earlier, if all vertices have some colour to choose from, the colouring is proper.

4. Let H be a graph, and n be an integer such that $n > |V(H)|$. Suppose $ex(n, H) = t$, and $tk > n^2 \log n$.

   Show that there is a k-colouring of edges of K_n so that there is no monochromatic copy of H.

   Solution: Since $ex(n, H) = t$, there exists a graph $G = ([n], E)$ that does not contain H as a subgraph and $|E| = t$. We will use G as a "mask" to colour K_n. To be more precise, for each colour c_i, we will choose a random permutation π_n of $[n]$, and colour (i, j) with c_i if $(\pi(i), \pi(j)) \in E$. We will do this sequentially from $i = 1$ to k. In each round i, we draw the random permutation, and colour edges accordingly with c_i. If we met an edge that has already been coloured, we just overwrite its original colour. Once this process is finished, every monochromatic graph of K_n is a subgraph of G, and thus does not contain H as a subgraph.

   The only problem is that the colouring process above may fail; that is, some edge may not receive any colour at all in the end of the process. What we are going to show next is that with strictly positive probability, the process does not fail. Namely, with positive probability, all edges will receive an colour.

   For a particular edge $e = (i, j)$ in K_n, let $X_{e,f}$ be the indicator variable of the event that $(\pi(i), \pi(j)) = f$ for some edge $f \in E$. Since the graph is undirected,

$$\mathbb{E}\, X_{e,f} = \Pr(X_{e,f} = 1) = \frac{1}{n} \cdot \frac{1}{n-1} + \frac{1}{n} \cdot \frac{1}{n-1} = \frac{2}{n(n-1)}.$$

   Let X_e be the indicator variable of the event that $(\pi(i), \pi(j)) \in E$. Then $X_e = \sum_{f \in E} X_{e,f}$ and by linearity,

$$\Pr(X_e = 1) = \mathbb{E}\, X_e = \sum_{f \in E} \mathbb{E}\, X_{e,f} = \frac{2t}{n(n-1)},$$

   where recall that $|E| = t$. Let A_e be the event that a particular edge e does not receive a colour at the end of the process. Thus,

$$\begin{aligned}
\Pr(A_e) &= (1 - \Pr(X_e = 1))^k \\
&= \left(1 - \frac{2t}{n(n-1)}\right)^k \leq e^{-\frac{2tk}{n(n-1)}} \\
&< e^{-\frac{2n^2 \log n}{n(n-1)}} < \frac{1}{n^2},
\end{aligned}$$

where we used the assumption that $tk > n^2 \log n$. The probability that the process succeeds is

$$\begin{aligned}
\Pr\left(\bigwedge_{e \in K_n} \overline{A_e}\right) &= 1 - \Pr\left(\bigvee_{e \in K_n} A_e\right) \\
&\geq 1 - \sum_{e \in K_n} \Pr\left(A_e\right) && \text{by the union bound} \\
&> 1 - \binom{n}{2} \frac{1}{n^2} > 1/2 > 0.
\end{aligned}$$

Therefore, there exists a sequence of permutations such that every edge is chosen at least once, which induces a colouring of K_n such that there is no monochromatic H.