Advanced Combinatorics - 2016 Fall
Solutions to Exercise 8

Comments and corrections are welcome.

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1. Prove that there exists a two-edge-colouring of K_n with at most

$$\binom{n}{a}2^{1-\binom{a}{2}}$$

monochromatic K_a.

Solution: Consider a uniformly at random 2-edge-colouring of K_n. Let X be the (random) total number of monochromatic cliques of size a in K_n, and X_S be the indicator variable of the event that the set S is monochromatic. Then $X = \sum_{S,\ |S|=a} X_S$. Note that

$$\mathbb{E}\, X_S = \Pr(X_S = 1) = \frac{2}{2^{\binom{a}{2}}}.$$

By the linearity of expectations,

$$\mathbb{E}\, X = \sum_{S,\ |S|=a} \mathbb{E}\, X_S = \binom{n}{a}2^{1-\binom{a}{2}}.$$

Hence, there must exist a colouring such that $X \geq \mathbb{E}\, X = \binom{n}{a}2^{1-\binom{a}{2}}$.

2. Using the alteration method, prove that the Ramsey number $R(4, k)$ satisfies

$$R(4, k) \geq \Omega((k/\log k)^2)$$

Solution: Recall that in the class we have showed that For any integer n and $p \in (0, 1)$,

$$R(4, k) > n - \binom{n}{4}p^{\binom{4}{2}} - \binom{n}{k}(1-p)^{\binom{k}{2}}.$$

A rough estimate is to set $\binom{n}{4}p^{\binom{4}{2}} < n/4$ and $\binom{n}{k}(1-p)^{\binom{k}{2}} < n/4$ and then $R(4, k) > n/2$. Note that

$$\binom{n}{4}p^{\binom{4}{2}} < \frac{n^4}{24}p^6.$$

Hence we should set $p = n^{-1/2}$. For the other constraint we use once again $1-p < e^{-p}$ and $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$, which yields,

$$\binom{n}{k}(1-p)^{\binom{k}{2}} \le \left(\frac{ne}{k}\right)^k e^{-p\binom{k}{2}} = \left(\frac{ne}{k}\right)^k e^{-k(k-1)/2\sqrt{n}} \le \left(\frac{ne}{k}\right)^k e^{-k^2/4\sqrt{n}},$$

where we use a loose bound $\frac{k(k-1)}{2} \ge \frac{k^2}{4}$ to make the calculation easier. Our goal is to set the right hand side less than $n/4$. Equivalently, taking the logarithm, we want

$$k \log n + k - k \log k - \frac{k^2}{4\sqrt{n}} \le \log n - \log 4.$$

As hinted by the question, we should take $n = c \cdot \left(\frac{k}{\log k}\right)^2$. One can verify that this choice does the job if $c = 1/16$, and thus $R(4,k) \ge n/2 = \frac{1}{32}\left(\frac{k}{\log k}\right)^2$.

Another way of deriving $n = \Omega\left(\left(\frac{k}{\log k}\right)^2\right)$ is the following. First note that n should be larger than k, and thus $k \log k$, k, and $\log n$ are of lower order of magnitude comparing to $k \log n$ and $\frac{k^2}{4\sqrt{n}}$. Thus all we need to do is to ensure that

$$\begin{aligned} \frac{k^2}{4\sqrt{n}} &\ge k \log n \\ \Leftrightarrow\ 4\sqrt{n}\log n &\le k, \end{aligned}$$

which indicates that we should set $n = \Omega\left(\left(\frac{k}{\log k}\right)^2\right)$.

3. An subset S of vertices in a hypergraph $H = (V, E)$ is *independent* if there is no $e \in E$ such that $e \subseteq S$. In other words, S does not completely contain any (hyper-)edge.

   Prove that every 3-uniform hypergraph with n vertices and $m \ge n/3$ edges contains an independent set of size at least

$$\frac{2n^{3/2}}{3\sqrt{3m}}.$$

Solution: This is similar to the graph case in class. Let S be a random set by choosing each vertex with probability p independently. Let $X = |S|$ and Y be the number of occupied hyperedges. Let Y_e be the indicator variable that all vertices in a hyperedge e are chosen. Thus, $Y = \sum_{e \in E} Y_e$. For a fixed $e \in E$,

$$\mathbb{E}\, Y_e = \Pr(Y_e = 1) = p^3.$$

Due to linearity of expectations,

$$\mathbb{E}\, X = np,$$

whereas

$$\mathbb{E}\,Y = \sum_{e \in E} \mathbb{E}\,Y_e = mp^3.$$

Similar to the graph case, the alteration is that we remove one vertex of each hyperedge, leaving an independent set I. Then $|I| = X - Y$. Again, by linearity of expectations,

$$\mathbb{E}\,|I| = \mathbb{E}\,X - \mathbb{E}\,Y = np - mp^3.$$

We will set np and mp^3 to have the same order of magnitude. Thus, p should be set to $c \cdot \sqrt{n/m}$ for some constant c. Plugging it back in, we have

$$\mathbb{E}\,|I| = (c - c^3) \cdot \frac{n^{3/2}}{\sqrt{m}}.$$

Optimizing $c - c^3$, we get that $c = 1/\sqrt{3}$ and

$$\mathbb{E}\,|I| = \frac{2}{3\sqrt{3}} \cdot \frac{n^{3/2}}{\sqrt{m}}.$$

4. Let $G = (V, E)$ be a graph. Associate each $v \in V$ a list $S(v)$ of colours of size at least $10d$ for some $d \geq 1$. Moreover, suppose that for each $v \in C$ and $c \in S(v)$, there are at most d neighbours u of v such that $c \in S(u)$.

   Prove that there is a proper colouring of G assigning to each vertex v a colour from its list $S(v)$.

   Solution: Consider a random colouring by assigning each v uniformly and independently a colour $c \in S(v)$. For each edge $(u, v) \in E$, let A_{uv}^c be the "bad" event that u and v have the same colour c.

   What events are correlated with A_{uv}^c? There are three possibilities (1) $A_{uv'}^{c'}$ where $v' \neq v$ but c' is arbitrary, (2) $A_{u'v}^{c'}$ where $u' \neq u$ but c' is arbitrary, or (3) $A_{uv}^{c'}$ where $c' \neq c$. The number of choices for the three cases are:

   (1) at most $10d$ choices for c' and d choices for v' by assumption, implying at most $10d^2$ choices in total;
   (2) same as case (1), at most $10d^2$ choices;
   (3) at most $(10d - 1)$ choices for c'.

   Thus, the total number of dependent events of any A_{uv}^c is at most $\Delta = 20d^2 + 10d \leq 30d^2$.

   The probability of A_{uv}^c is $p = \Pr(A_{uv}^c) = \left(\frac{1}{10d}\right)^2 = \frac{1}{100d^2}$. We want to apply the symmetric version of the Lovász Local Lemma. It only needs to verify that $ep\Delta = e \cdot 30d^2 \cdot \frac{1}{100d^2} < 1$. Hence, there is a colouring such that none of A_{uv}^c holds, namely a proper colouring.

5. Let $G = (V, E)$ be a cycle of length $11n$, and $V = V_1 \cup V_2 \cup \cdots \cup V_n$ be an arbitrary partition of its $11n$ vertices; that is, $V_i \cap V_j = \emptyset$ for any $1 \leq i \neq j \leq n$. Moreover, $|V_i| = 11$ for every $i \in [n]$.

   Prove that there exists an independent set of G that contains precisely one vertex from each V_i.

   Solution: Uniformly at random chooce one vertex from each V_i. For each $(u, v) \in E$ and u, v do not belong to the same V_i, let A_{uv} be the event that both u and v are chosen. Clearly $\Pr(A_{uv}) = \frac{1}{11^2} = \frac{1}{121}$.

   We want to apply the symmetric version of the Lovász Local Lemma. How many events are correlated with A_{uv}? Since G is a cycle, there are at most two events having the form $A_{uu'}$ and $A_{vv'}$. Another possibility is A_{xy} where x or y is in the same partition as u or v. There are at most 10 choices for the vertex and there are two events associated with the vertex, implying at most 20 possibilities for each u and v, and 40 possibilities in total. Hence, the maximum degree Δ in the dependency graph is at most $2 + 40 = 42$. We may verify that

$$ep(\Delta + 1) = \frac{43e}{121} < 1.$$

   The condition of symmetric LLL is met, and there must exist an independent set that contains precisely one vertex from each V_i.