# Sampling Proper Colorings on Line Graphs Using $(1 + o(1))\Delta$ Colors


Chihao Zhang

Shanghai Jiao Tong University


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Joint work with Yulin Wang (SJTU) and Zihan Zhang (SJTU $\to$ NII)

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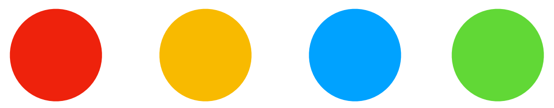
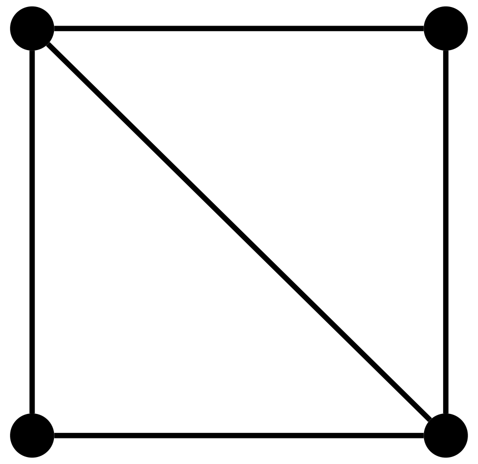
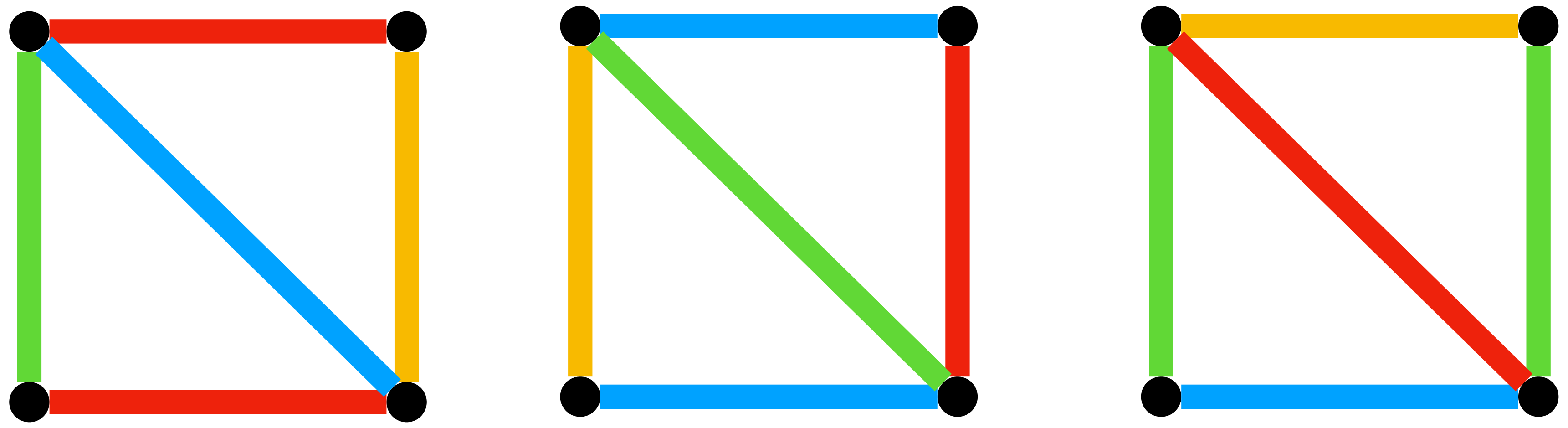
Color List $L = \{$  $\}$

$G =$ 



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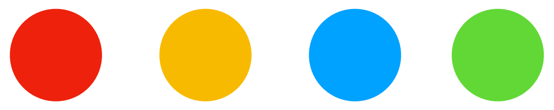
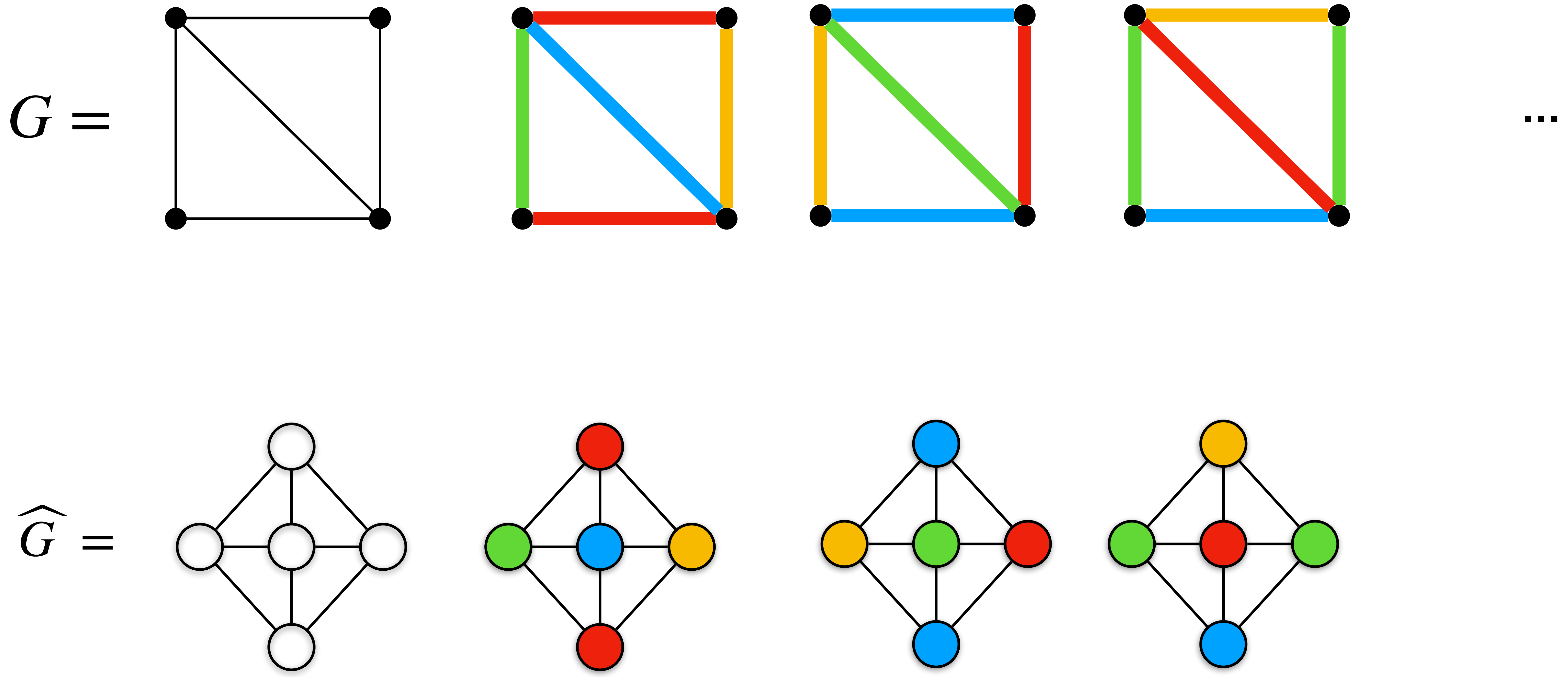
Color List $L = \{$  $\}$



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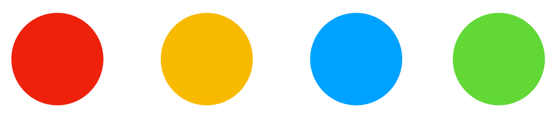
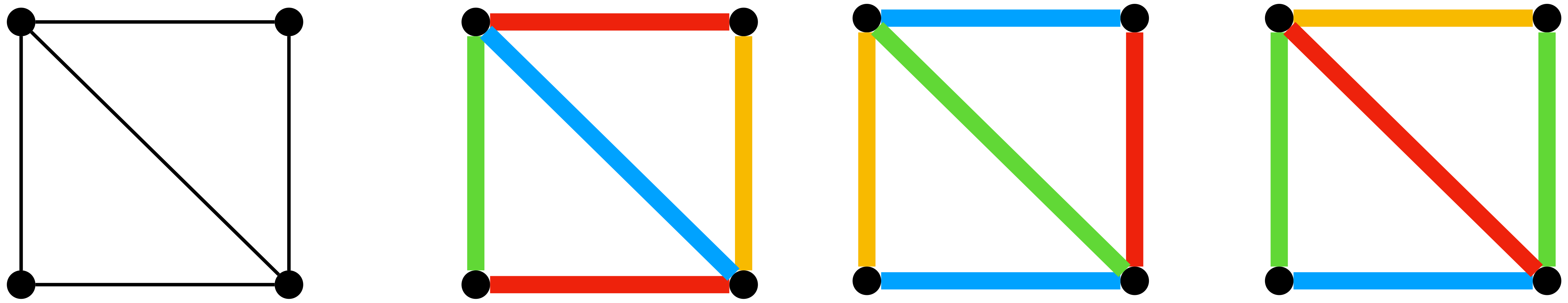
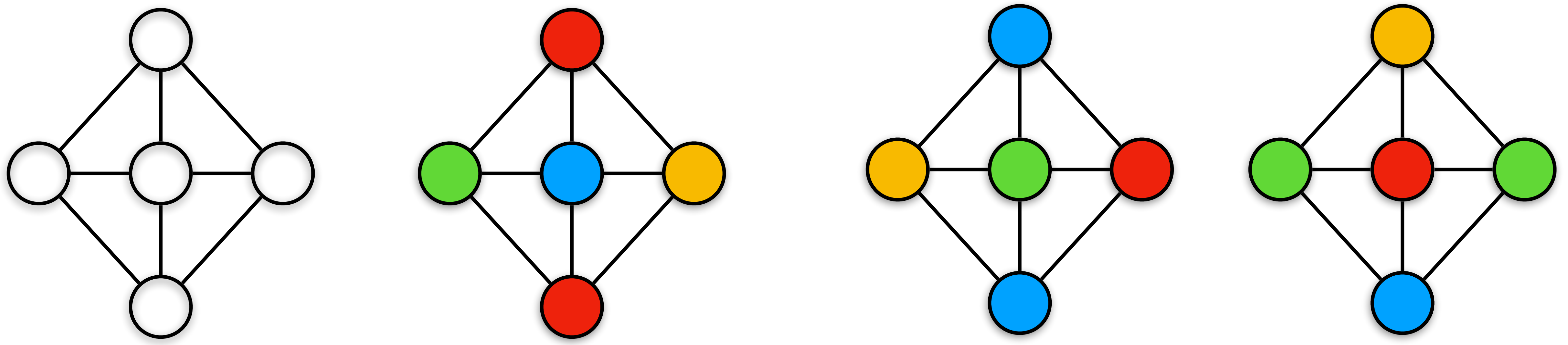
Color List $L = \{$  $\}$

$G =$  ...

$\widehat{G} =$ 

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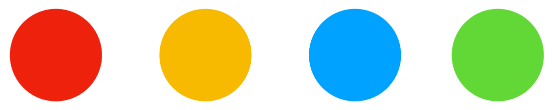
Color List $L = \{$  $\}$

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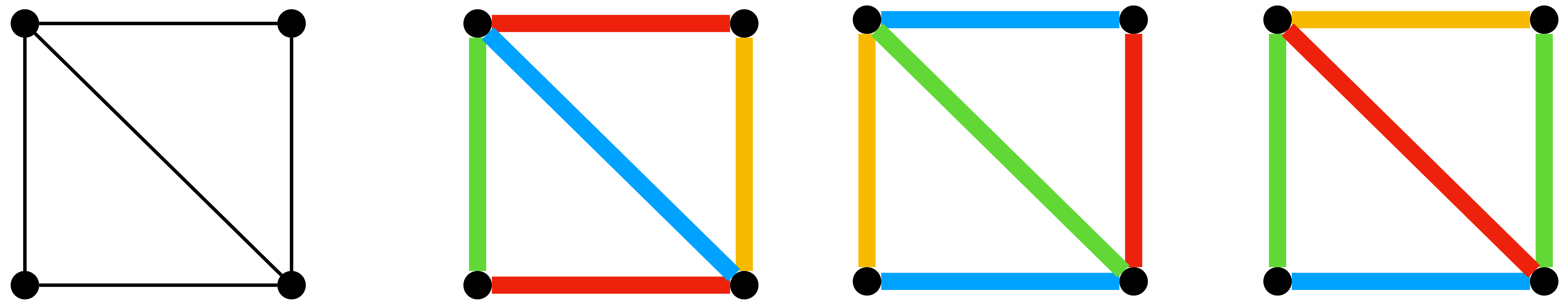

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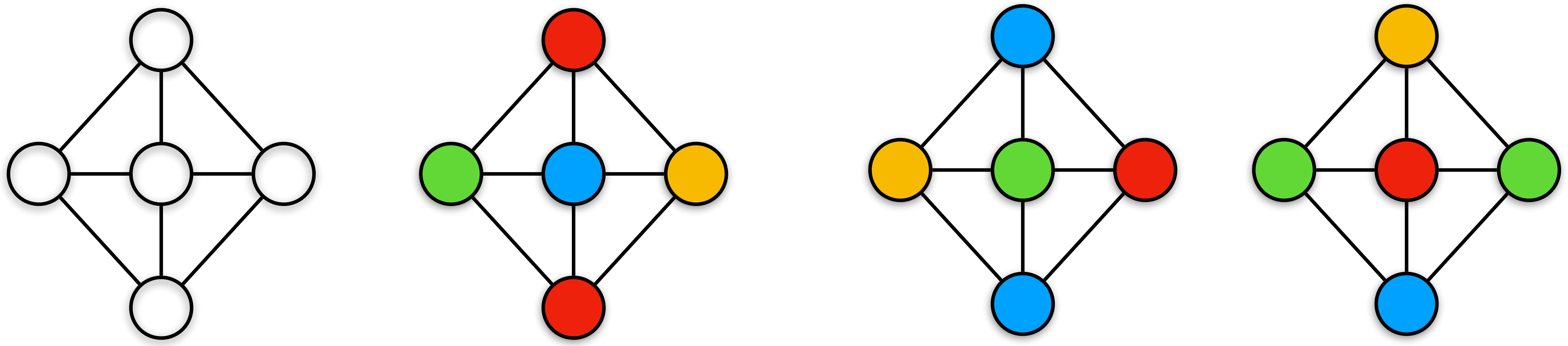

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proper edge coloring on $G \iff$ proper vertex coloring on $\widehat{G}$

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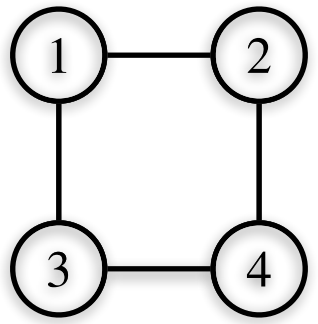
$G =$ 

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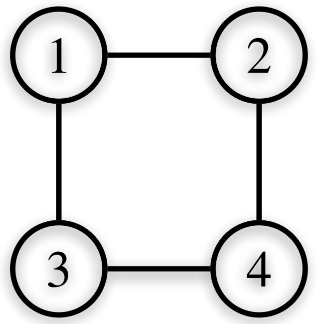
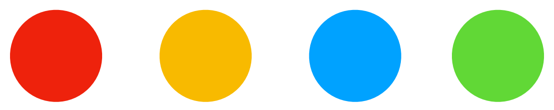
$G =$ 


Color List $L = \{$  $\}$

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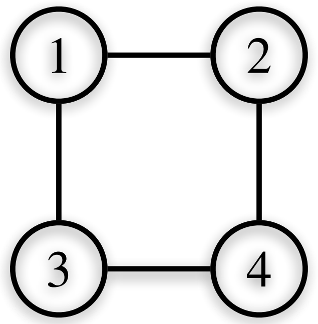
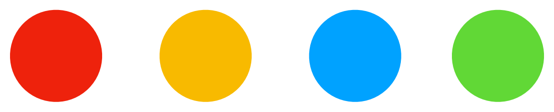
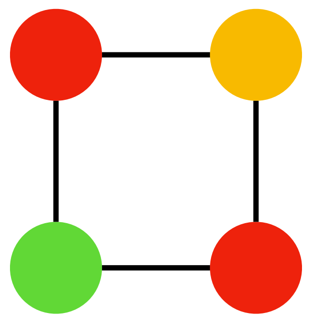
$G =$ 


Color List $L = \{$  $\}$



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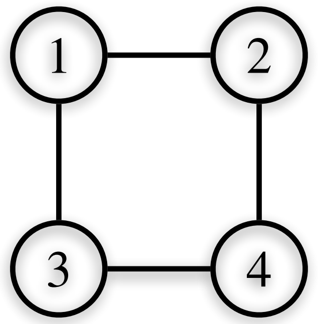
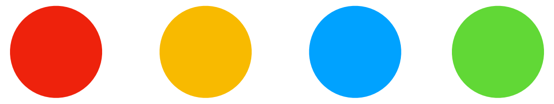
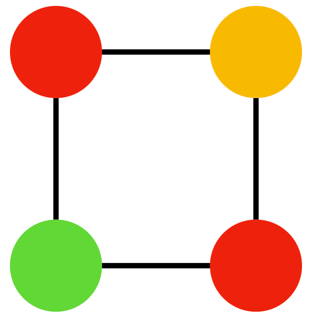
$G =$ 


Color List $L = \{$  $\}$



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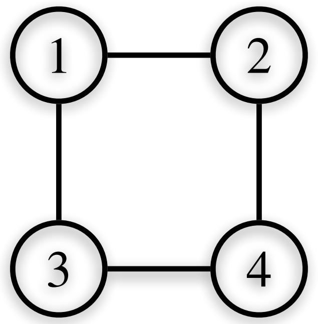
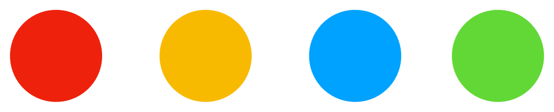
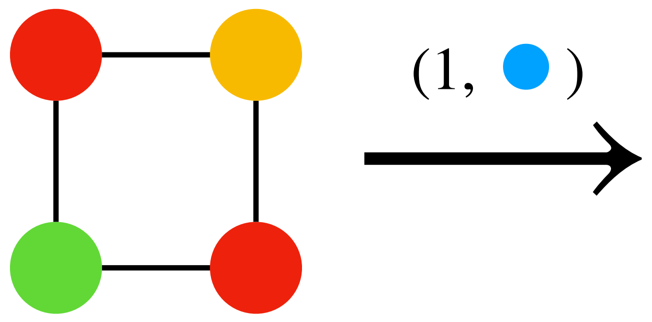
$G =$ 


Color List $L = \{$  $\}$



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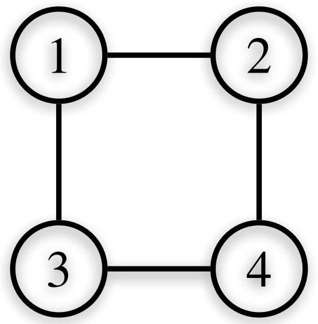
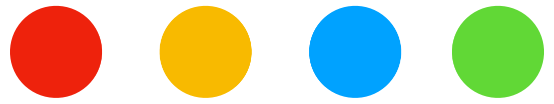
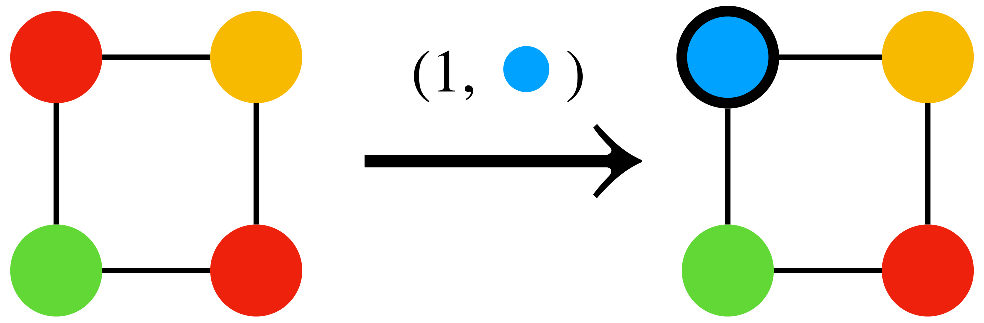
$G =$ 


Color List $L = \{$  $\}$



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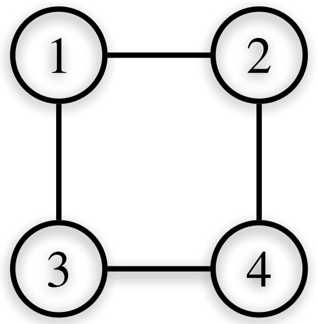
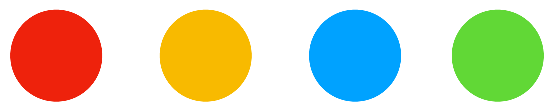
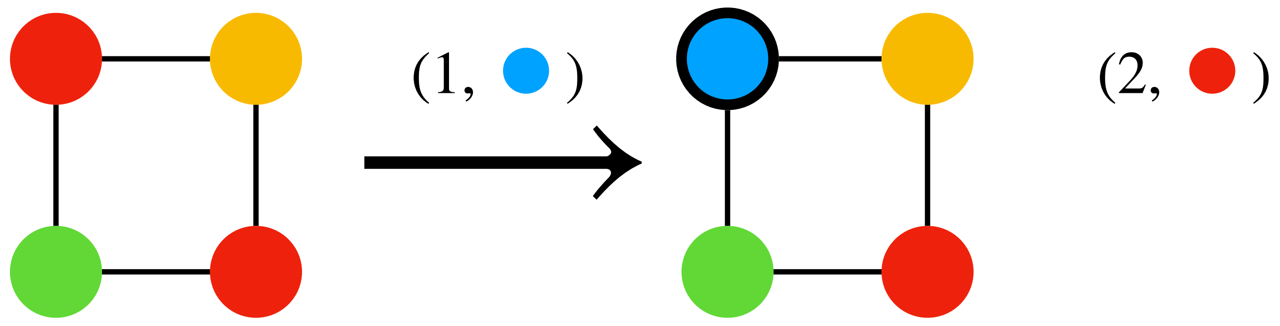
$G =$ 


Color List $L = \{$  $\}$



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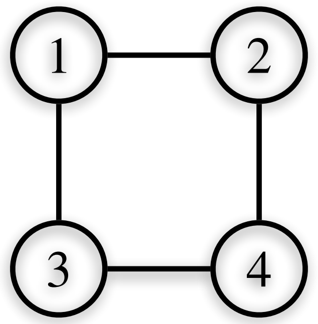
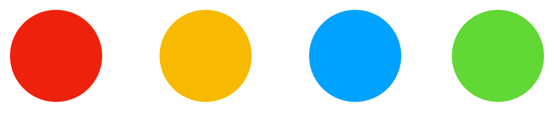
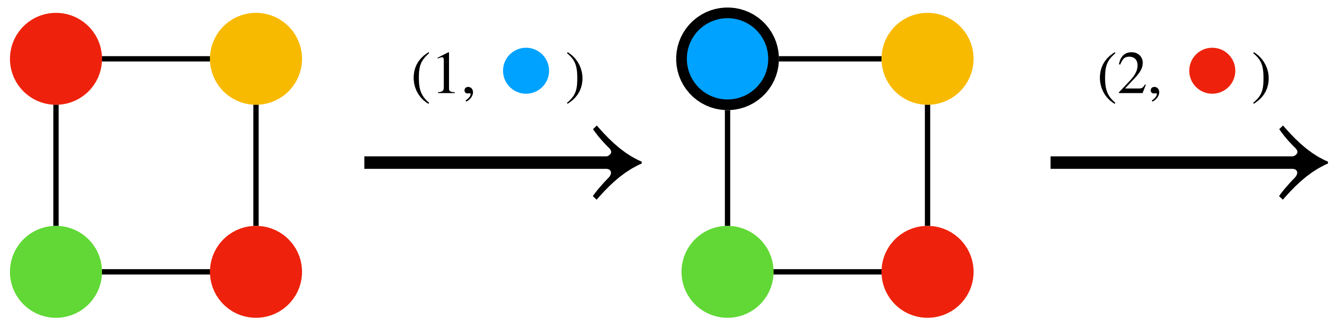
$G =$ 


Color List $L = \{$  $\}$



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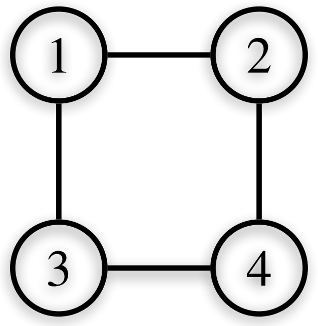
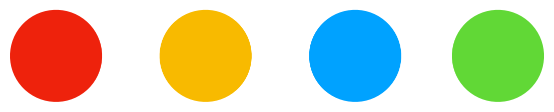
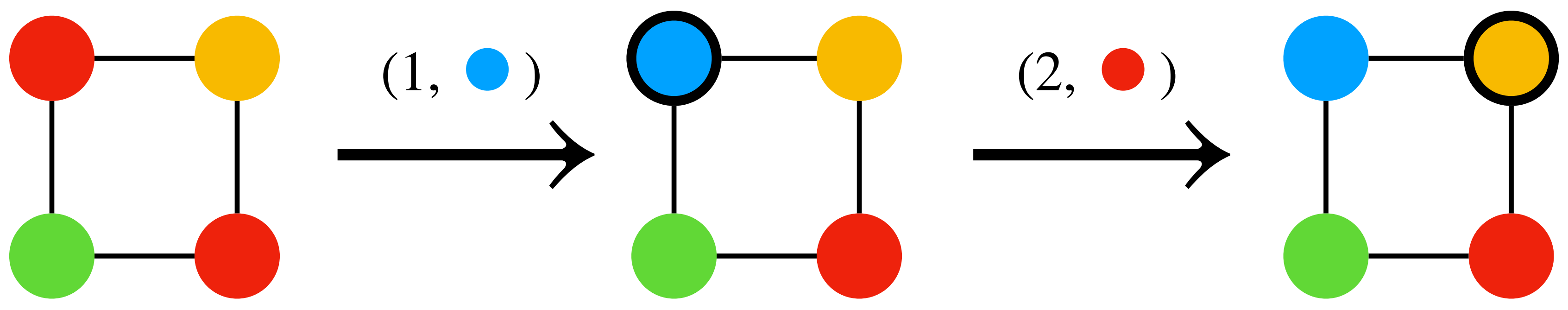
$G =$ 


Color List $L = \{$  $\}$



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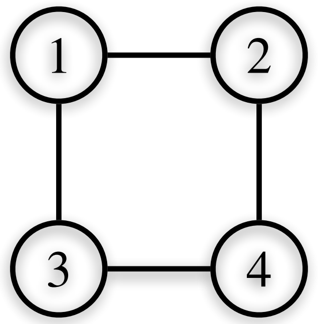
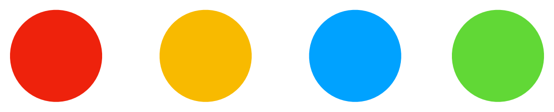
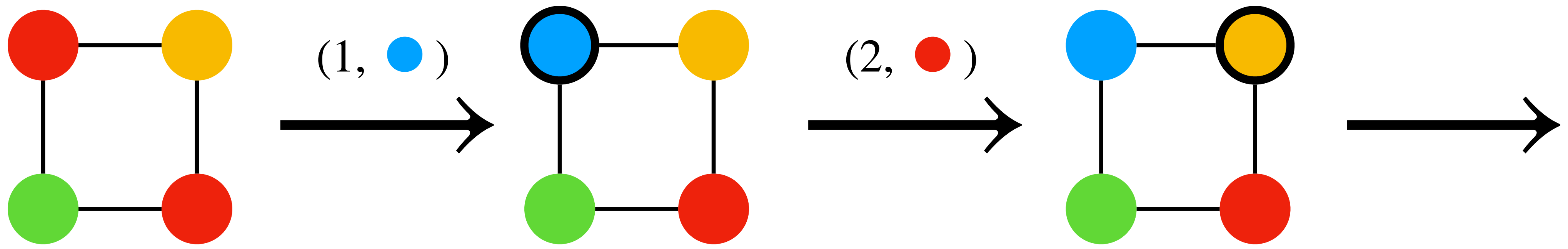
$G =$ 


Color List $L = \{$  $\}$



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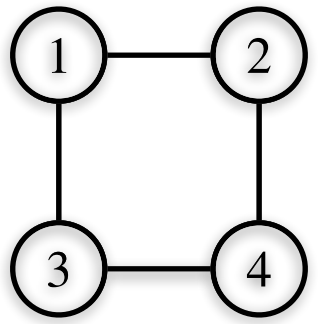
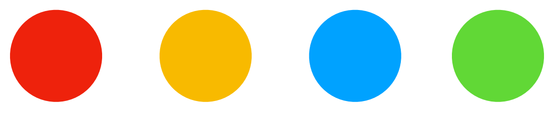
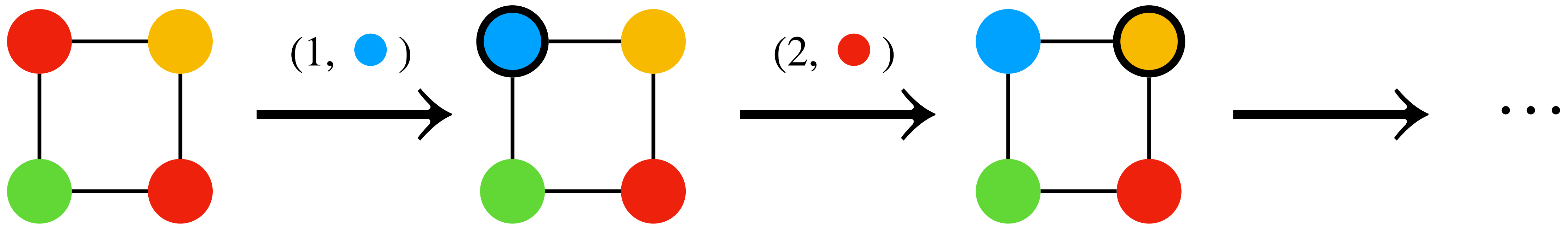
$G =$ 


Color List $L = \{$  $\}$



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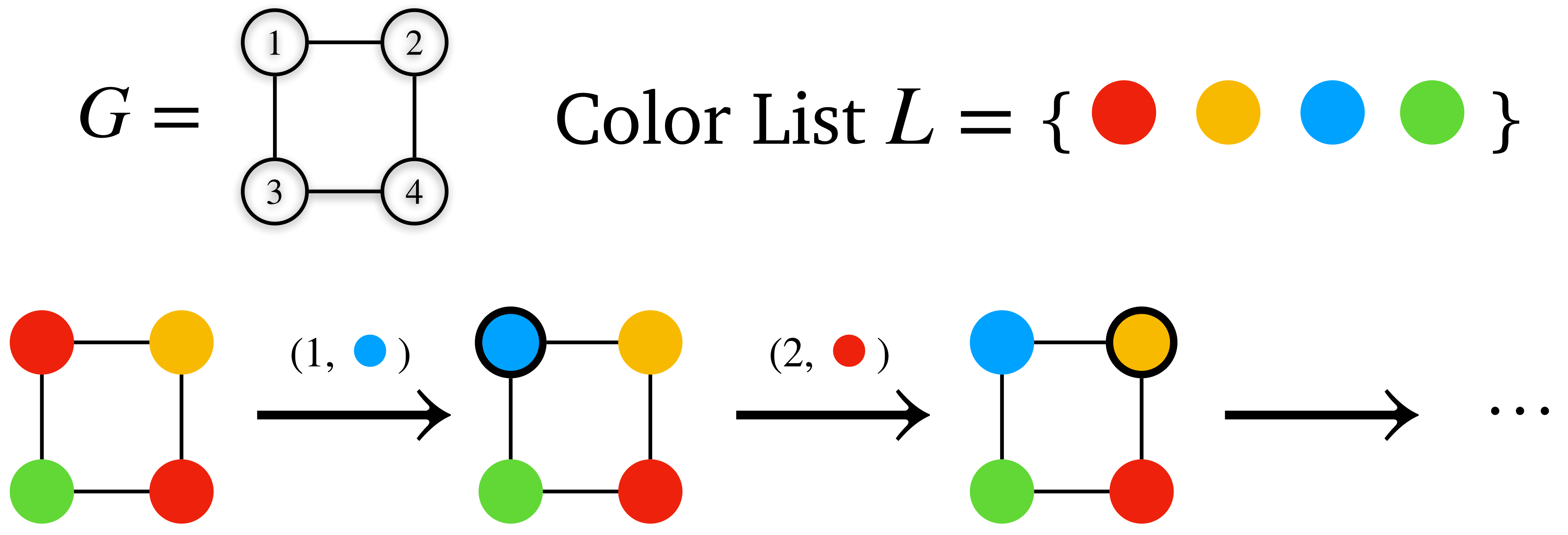


($q$: number of colors; $\Delta$: maximum degree)

**Conjecture:** The chain is rapidly mixing when $q \geq \Delta + 2$

## Condition for rapid mixing

- $q > \left(\frac{11}{6} - \epsilon\right)\Delta$ in general [CDMPP'19]
- $q > \frac{10}{6}\Delta$ for line graphs [ALOG'21]
- $q > (1 + o(1))\Delta$ for line graphs [this work]

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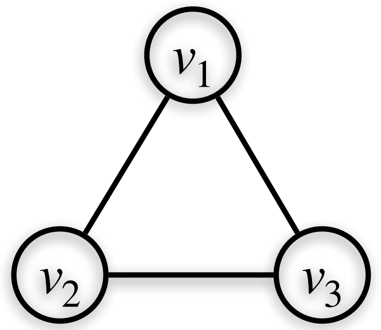
$G =$ 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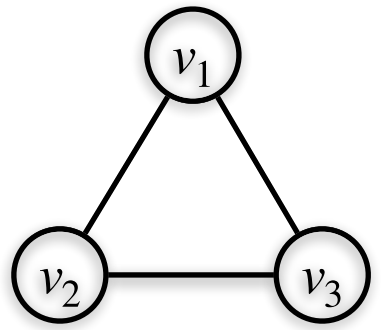
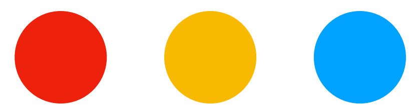
$G =$ 


Color List $L = \{$  $\}$

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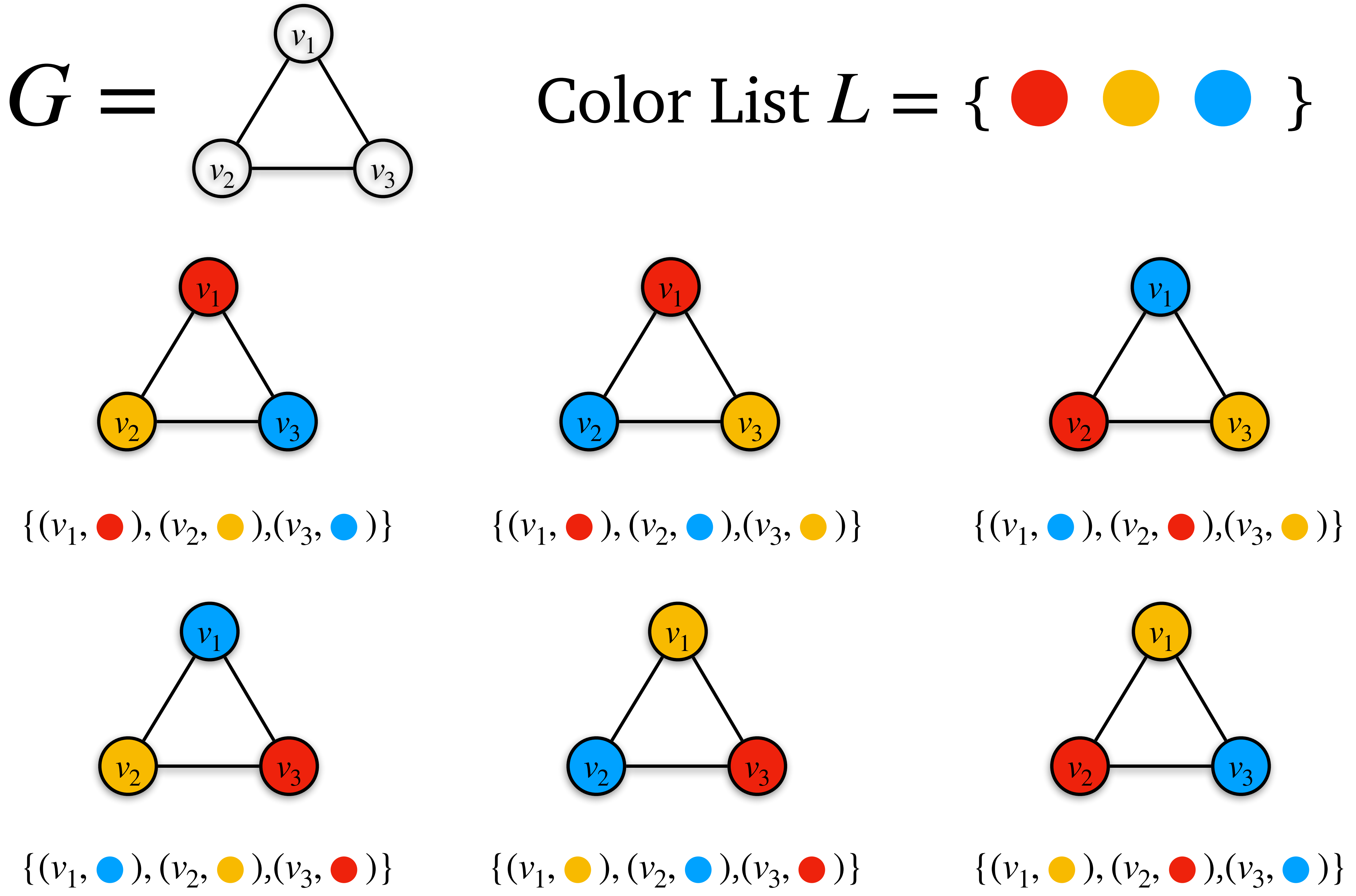

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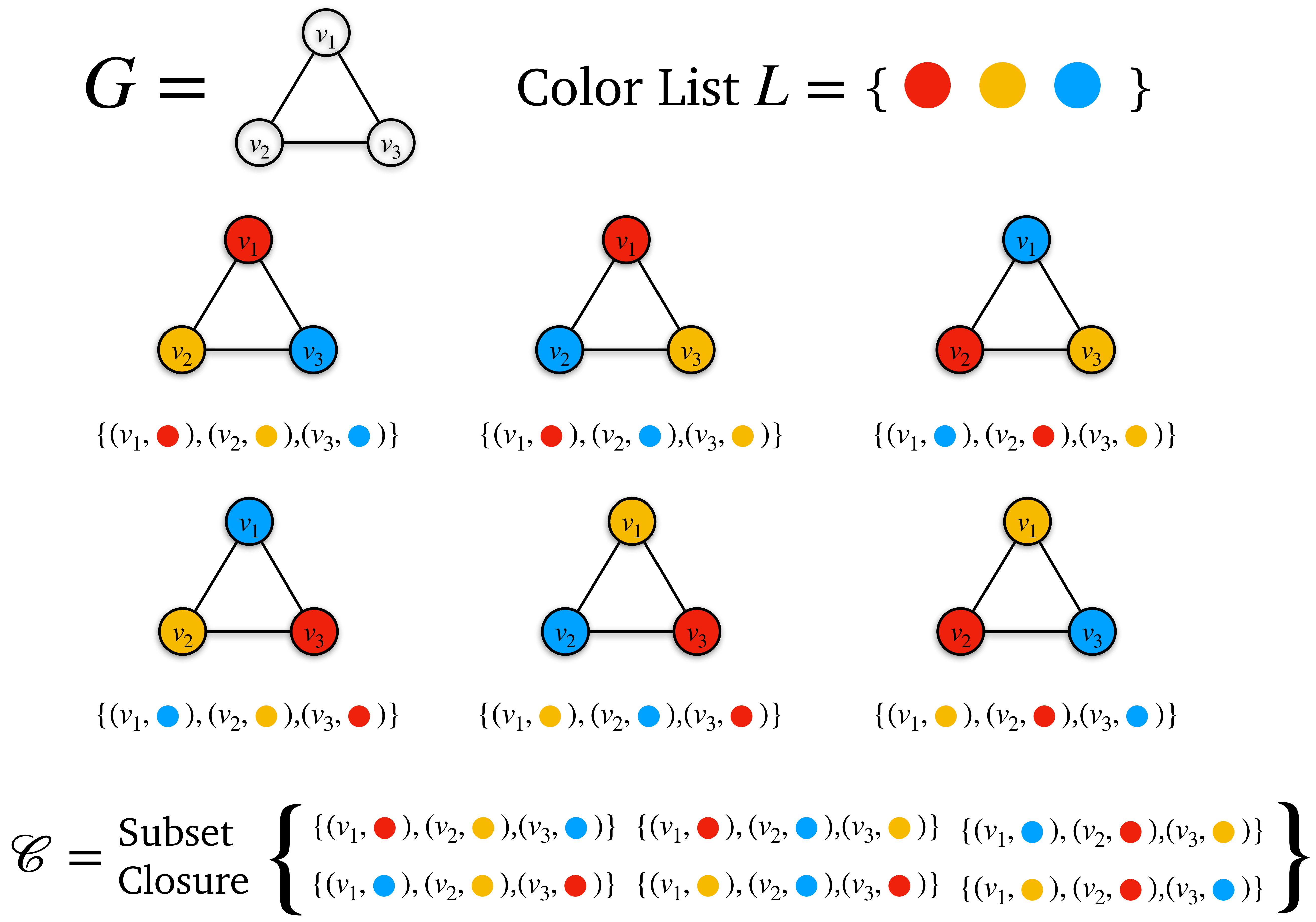

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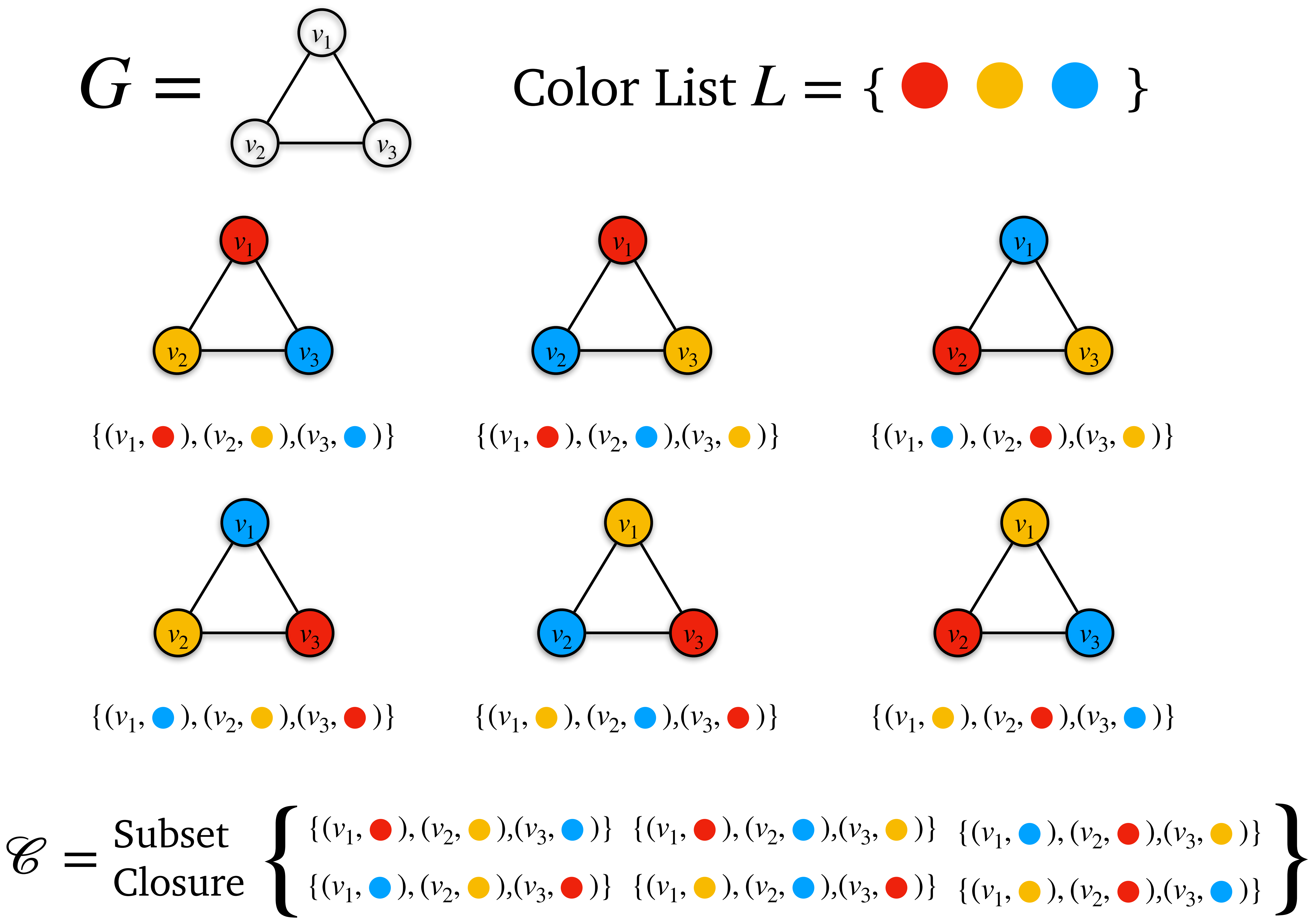

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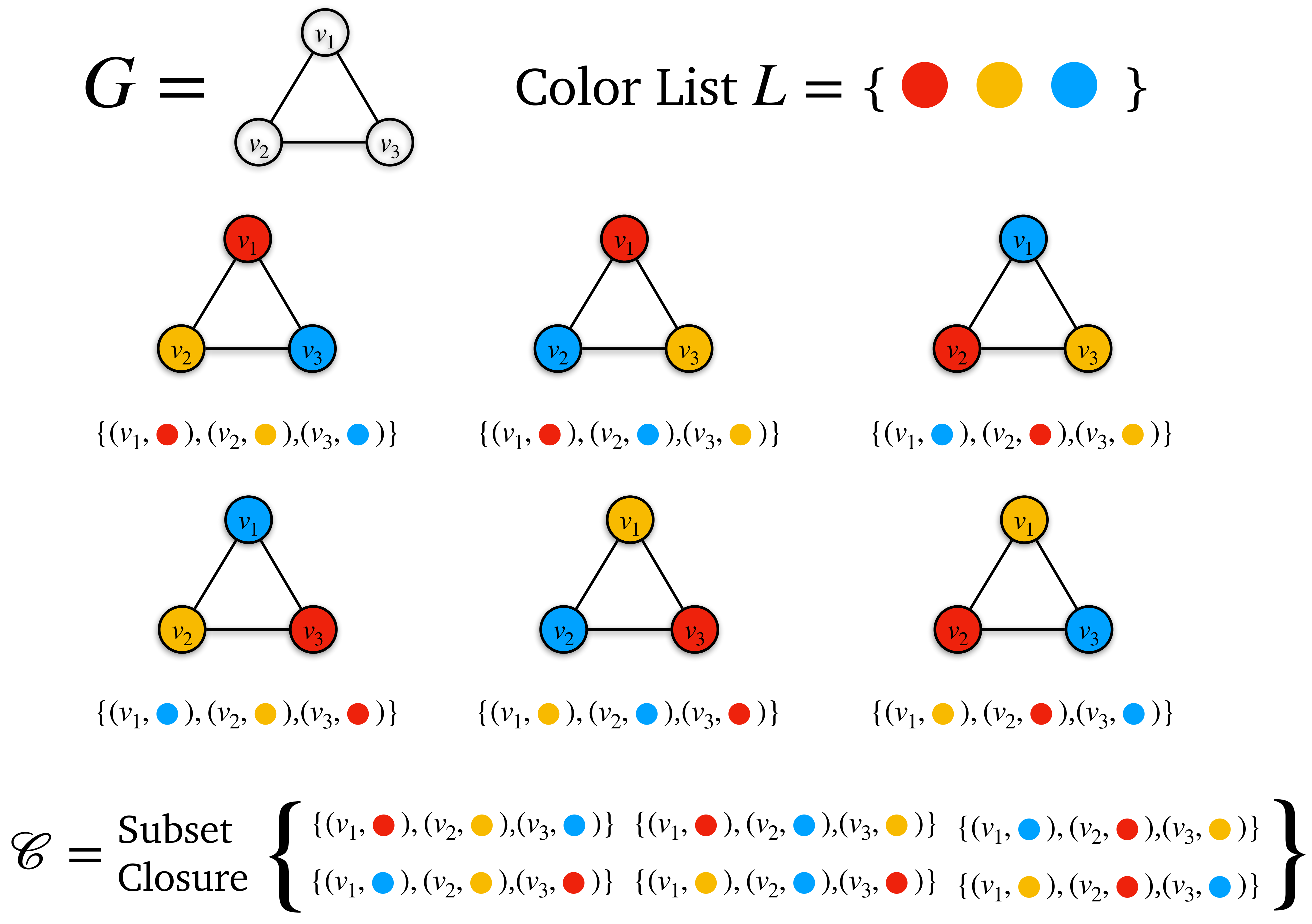
*collection of all partial colorings*



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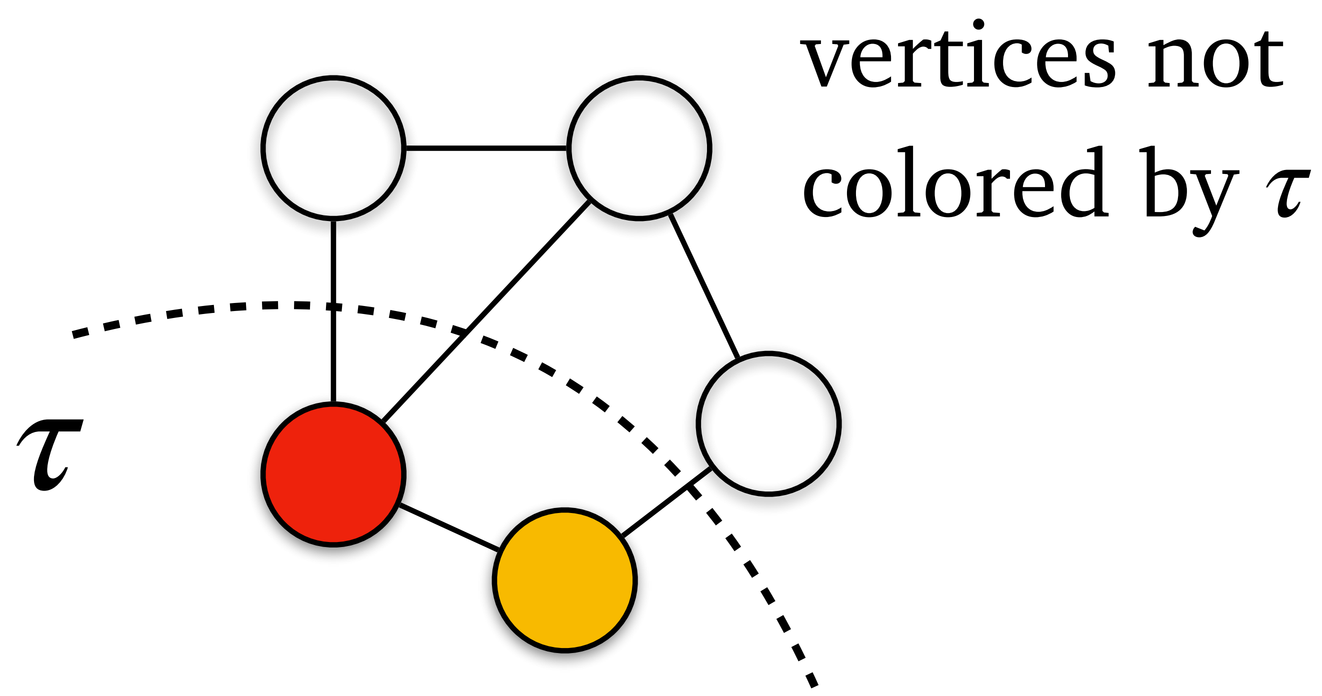

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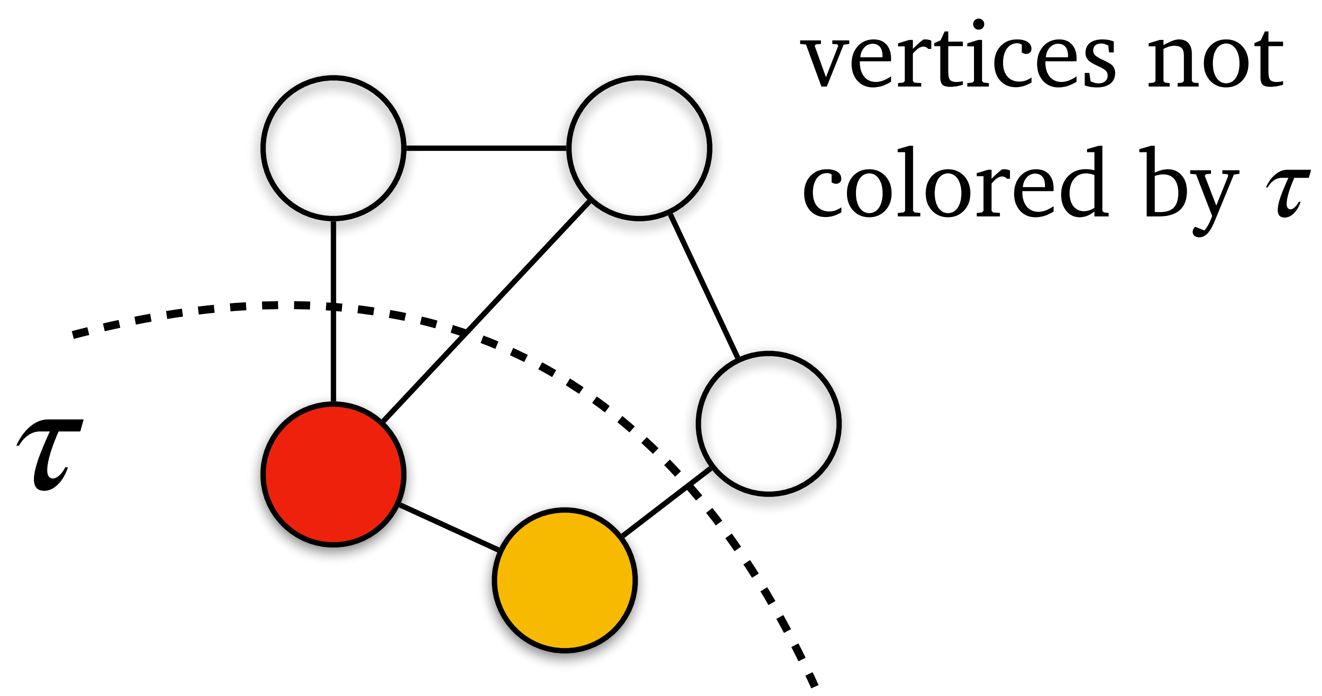
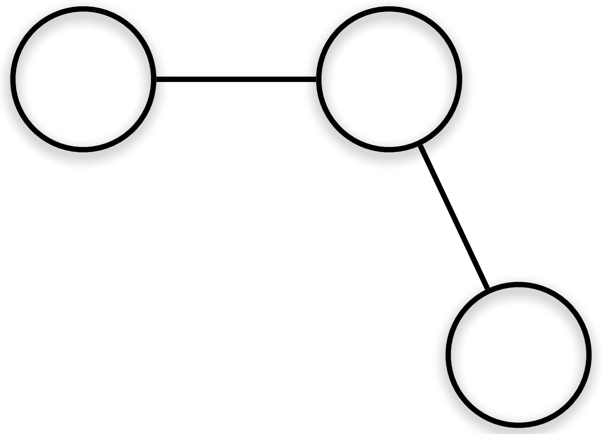



$\mathscr{C}_\tau$: proper colorings on



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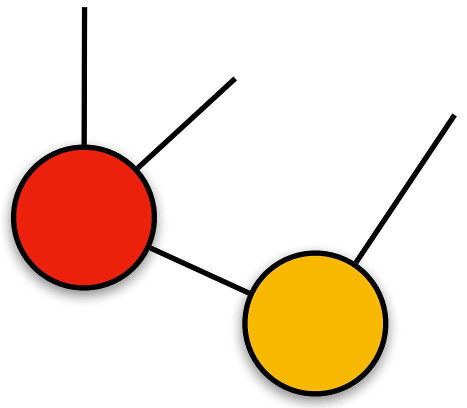

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The Glauber dynamics mixes rapidly on line graphs when $q > (1+o(1))\Delta$

# Matrix Trickle-Down Theorem

Abdolazimi, Liu and Oveis Gharan established the following theorem:

- The local walk on $\mathscr{C}$ is irreducible
- For a family of matrices $\left\{N_x \in \mathbb{R}^{\mathscr{C}(1)\times\mathscr{C}(1)}\right\}$ and $\alpha \geq \frac{1}{2}$

$$P_x - \alpha \mathbf{1}\pi_x^\top \preceq_{\pi_x} N_x \preceq_{\pi_x} \frac{1}{2\alpha+1}\mathtt{Id}$$

- $\mathop{\mathbf{E}}_{x\sim\pi}\left[\Pi_x N_x\right] \preceq \Pi N - \alpha \Pi N^2$

Then $P - \left(2 - \frac{1}{\alpha}\right)\mathbf{1}\pi^\top \preceq_\pi N$

- $\pi_x$ stationary distr. of $P_x$
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- $\Pi_x = \mathrm{diag}(\pi_x)$
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# MTD on Spin System

Suppose a family of matrices $\{M_\tau\}$ satisfies

- For every $\tau \in \mathscr{C}(n-2)$: $\Pi_\tau P_\tau - 2\pi_\tau\pi_\tau^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_\tau$
- For every $\tau \in \mathscr{C}(n-k)$ with $k > 2$ such that $\mathscr{C}_\tau$ is connected:

$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_\tau \text{ and } \mathop{\mathbf{E}}_{x\sim\pi_\tau}[M_{\tau\cup\{x\}}] \preceq M_\tau - \frac{k-1}{k-2}M_\tau\Pi_\tau^{-1}M_\tau$$

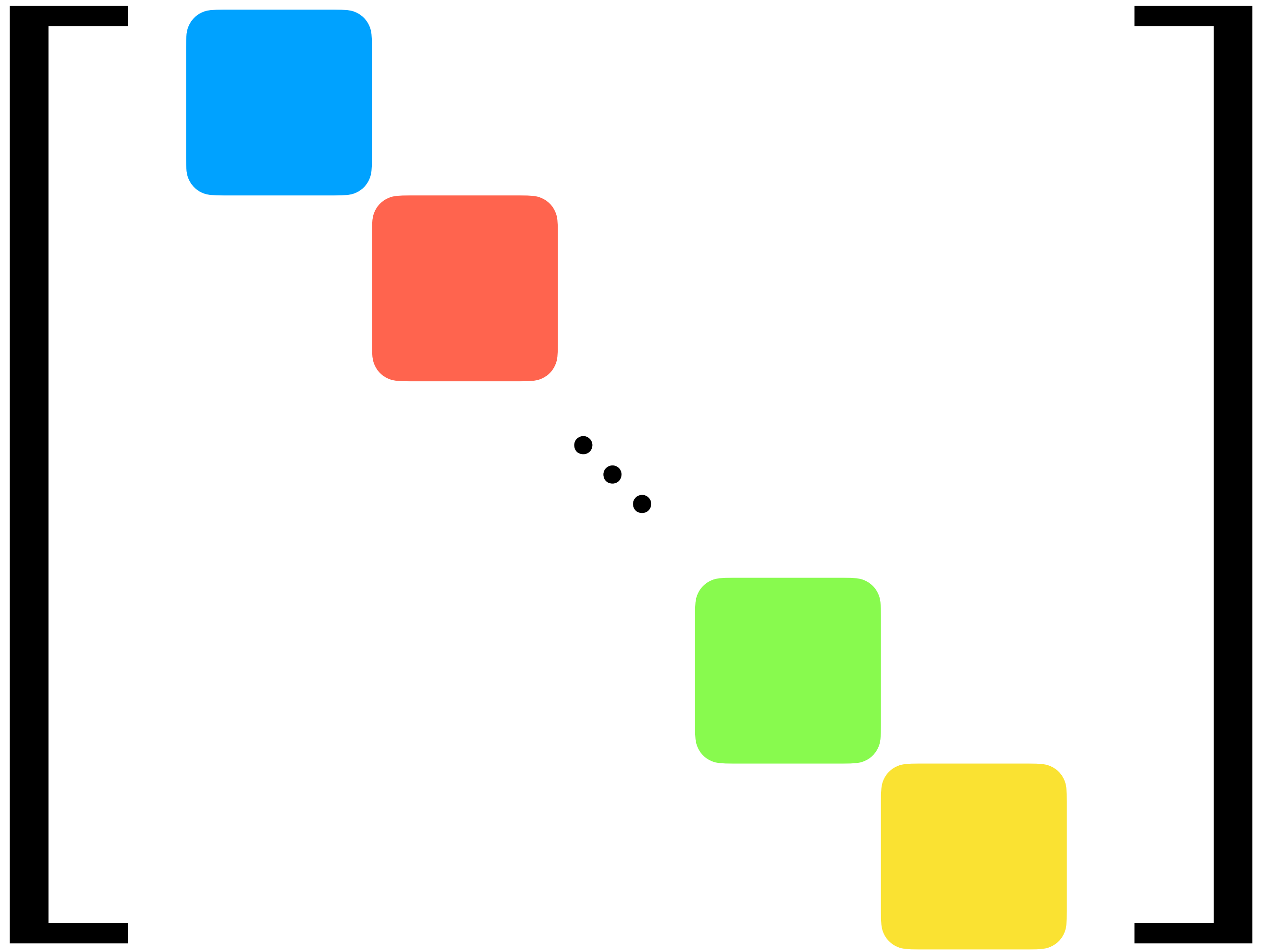
Then for every $\tau \in \mathscr{C}(n-2)$: $\lambda_2(P_\tau) \leq \lambda_1(\Pi^{-1}M_\tau)$

# Our Construction of $M_\tau$

For each color $c$, there is a matrix $M_\tau^c$

$M_\tau$ is a block-diagonal matrix with each $M_\tau^c$ on its diagonal

$M_\tau =$



In our construction, $M_\tau^c$ is only supported on those $(uc, vc)$ with $u \sim v$

# The Base Case

goal: $\beta = o(\Delta)$

Assume each vertex $v$ has $\deg(v) + \beta$ colors

The base case is when $\tau \in \mathscr{C}(n-2)$

We can directly compute $P_\tau - 2\mathbf{1}\pi_\tau^\top$
and pick $M_\tau^c$ so that its nonzero entries
are approximately

$$\Pi_\tau \begin{bmatrix} \frac{1}{\beta^2} & -\frac{1}{\beta} \\ -\frac{1}{\beta} & \frac{1}{\beta^2} \end{bmatrix}$$

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The inductive constraint is $\mathop{\mathbf{E}}_{x\sim\pi_\tau}[M^c_{\tau\cup\{x\}}] \preceq M^c_\tau - \frac{k-1}{k-2} M^c_\tau \Pi_\tau^{-1} M^c_\tau$

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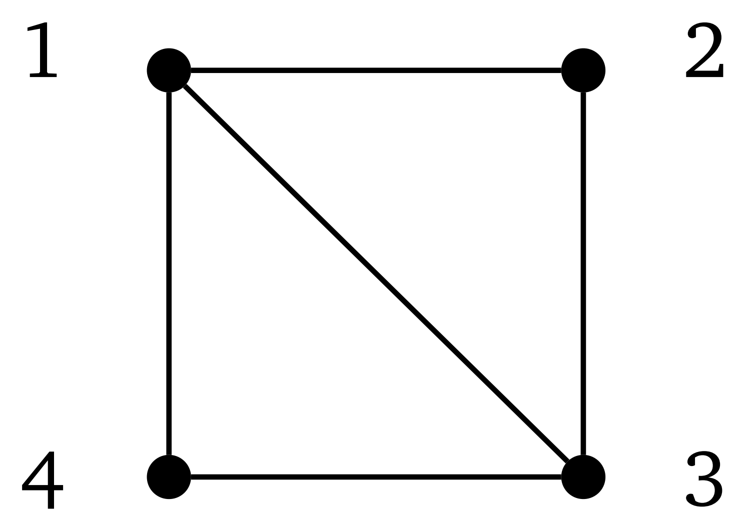

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Goal: Upper bound [blue box] by a diagonal matrix so that it becomes to a scalar inequality

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Bound by the spectrum of $K_\Delta$

trivially bound by row sum

# The Scalar Constraints

The system reduces to a set of scalar constraints

$$\begin{cases} b_1 \leq \frac{1}{\beta^2} \\ (h-1)b_h - h \cdot b_{h-1} \geq C_1 b_h^2 + \frac{C_2}{\beta^2} h^{2\alpha}, \quad 2 \leq h \leq H \end{cases}$$

**Proposition.** For any $1/2 \leq \alpha \leq 1$, the system has solution when $\beta \geq cH^\alpha \log^2 H$ for some constant $c > 0$.

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