Reversible random walks on *dynamic* graphs

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Contents

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Reversible random walks on dynamic graphs

Random Structures & Algorithms, 2023

Main topic. Time-**in**homogeneous Markov chain

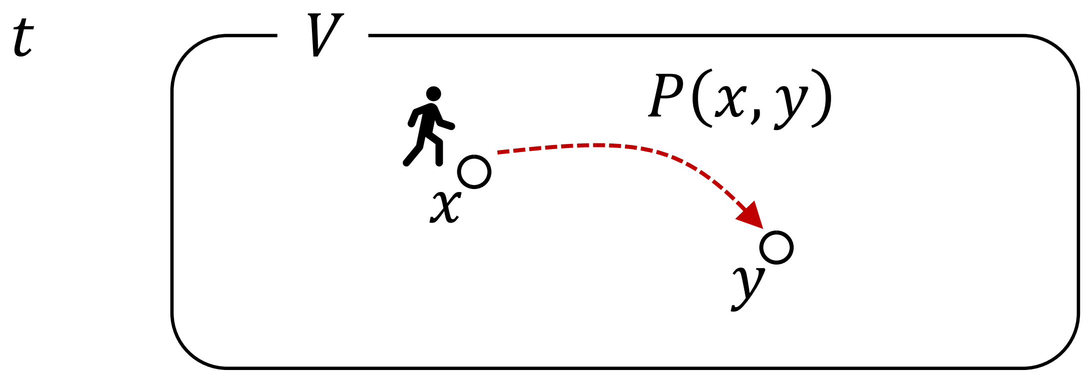
- Notation and Main result
- Previous work
- Idea of proof
- Other topic

Notation: Random walk

- ✓ V: Set of n vertices
- ✓ $P \in [0,1]^{V \times V}$: Transition matrix on V

At each discrete time step $t = 1,2, \ldots,$

the walker moves from x to y with probability $P(x, y)$



Referred to as the **random walk according to P**

- ✓ i.e., time-homogeneous Markov chain

Notation: Random walk

- V: Set of n vertices
- $P \in [0,1]^{V \times V}$: Transition matrix on V

<u>Random walk according to P</u>.

A sequence of random variables $X_0, X_1, X_2, \ldots$ s.t.

(Markov property)

$$\Pr\left(X_t = x_t \,\middle|\, \begin{array}{c} X_0 = x_0, \\ \vdots \\ X_{t-1} = x_{t-1} \end{array} \right) = \Pr(X_t = x_t \mid X_{t-1} = x_{t-1}) = P(x_{t-1}, x_t)$$

holds for all $t = 1,2, \ldots$ and $(x_0, \ldots, x_t) \in V^{t+1}$

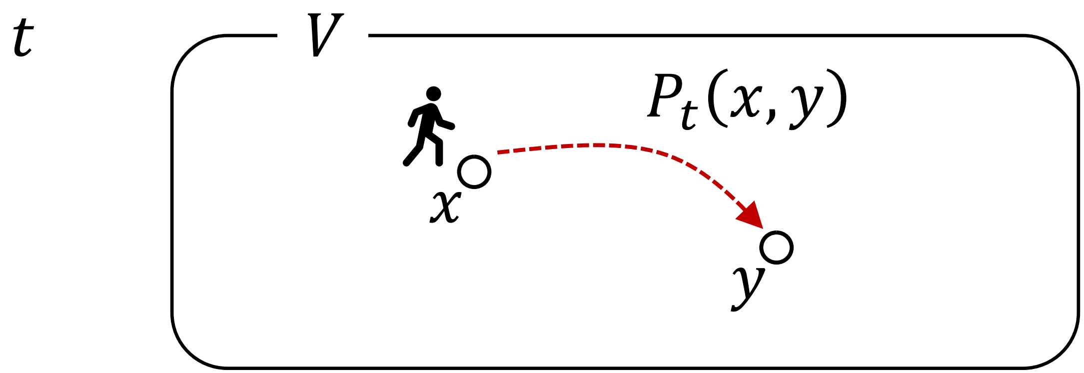
(where $\Pr(X_0 = x_0, \ldots, X_{t-1} = x_{t-1}) > 0$)

- i.e., time-homogeneous Markov chain

Notation: **Random walk according to $(P_t)_{t\geq 1}$**

- ✓ V: Set of n vertices
- ✓ $(P_t)_{t\geq 1} = (P_1, P_2, \ldots)$: Sequence of transition matrices on V

At each discrete time step $t = 1,2, \ldots,$
the walker moves from x to y with probability $\boldsymbol{P_t(x,y)}$



Referred to as the **random walk according to $(P_t)_{t\geq 1}$**

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  - ➢ Transition matrix at time t is P_t

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Random walk according to $(P_t)_{t\geq 1}$.

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(Markov property)

$$\Pr\left(X_t = x_t \;\middle|\; \begin{matrix} X_0 = x_0, \\ \vdots \\ X_{t-1} = x_{t-1} \end{matrix} \right) = \Pr(X_t = x_t \mid X_{t-1} = x_{t-1}) = P_t(x_{t-1}, x_t)$$

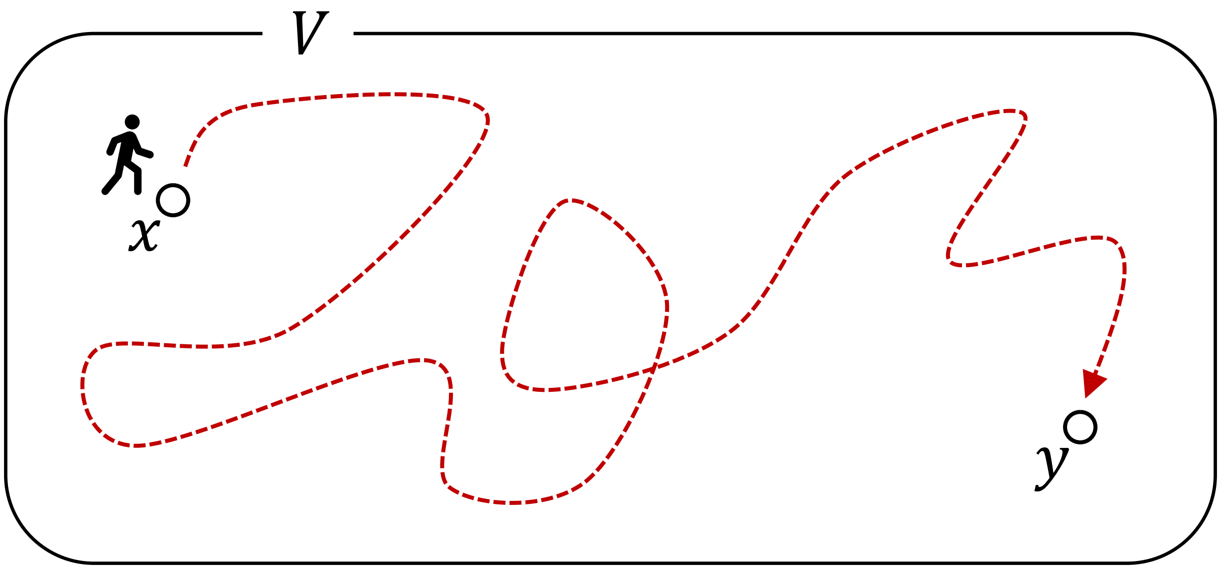
holds for all $t = 1,2,\dots$ and $(x_0, \dots, x_t) \in V^{t+1}$

(where $\Pr(X_0 = x_0, \dots, X_{t-1} = x_{t-1}) > 0$)

- ✓ i.e., time-**in**homogeneous Markov chain
  - ➢ Transition matrix at time t is P_t

Hitting time $\boldsymbol{t}_{\mathbf{hit}} \coloneqq \max_{x,y \in V} \mathbf{E}_x[\min\{\, t \geq 0 \mid X_t = y \,\}]$

✓ The expected # of steps for the walker to move from x to y (considering the worst-case pair of vertices x and y)



Hitting time

Hitting time $\boldsymbol{t}_{\mathbf{hit}} \coloneqq \max_{x,y \in V} \mathbf{E}_x[\min\{ t \geq 0 \mid X_t = y \}]$

✓ The expected # of steps for the walker to move from x to y (considering the worst-case pair of vertices x and y)

Write $\boldsymbol{t}_{\mathbf{hit}}(\boldsymbol{P})$ as the HT of the RW according to P

i.e., HT of a RW with a <u>time-**in**variant</u> transition matrix

✓ There is much previous work

Write $\boldsymbol{t}_{\mathbf{hit}}\left((\boldsymbol{P}_{\boldsymbol{t}})_{\boldsymbol{t} \geq \mathbf{1}}\right)$ as the HT of the RW according to $(P_t)_{t \geq 1}$

i.e., HT of a RW with <u>time-**varying**</u> transition matrices

✓ Not much is known (This work)

Main result (Hitting time)

✓ We give an upper bound on HT of a RW with time-varying transition matrices in terms of HTs of time-**in**variant ones:

Thm.1 (Hitting time). Suppose $(P_t)_{t\geq 1}$ satisfies the following:

✓ All $P_1, P_2, \ldots$ are irreducible, reversible, and lazy

✓ **All $P_1, P_2, \ldots$ have the same stationary distribution π**

Then, there is a constant $C > 0$ s.t.

$$\boldsymbol{t_{\mathrm{hit}}\big((P_t)_{t\geq 1}\big) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$$
$$= C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), t_{\mathrm{hit}}(P_3), \ldots\}.$$

$t_{\mathrm{hit}}\big((P_t)_{t\geq 1}\big)$: HT of the RW according to $(P_t)_{t\geq 1}$, i.e., HT of the RW with time-varying transition matrices (P_t at time t)

$t_{\mathrm{hit}}(P_1)$: HT of the RW according to P_1, i.e., HT of the RW with the time-invariant transition matrix (P_1 at all times)

Thm.1 (Hitting time). Suppose $(P_t)_{t\geq 1}$ satisfies the following:

✓ All $P_1, P_2, \ldots$ are irreducible, reversible, and lazy

✓ **All $\boldsymbol{P_1, P_2, \ldots}$ have the same stationary distribution $\boldsymbol{\pi}$**

Then, there is a constant $C > 0$ s.t.

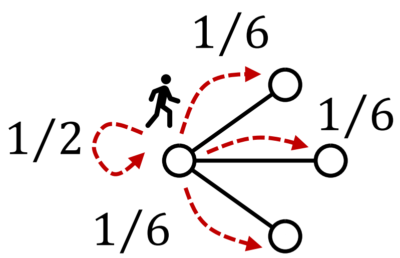
$$\boldsymbol{t_{\mathrm{hit}}\big((P_t)_{t\geq 1}\big) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$$
$$= C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), t_{\mathrm{hit}}(P_3), \ldots\}.$$

Previous work: lazy simple RW on a (*static*) graph

- ✓ G: n-vertex graph
  - $V(G)$: Vertex set of G
  - $E(G)$: Edge set of G
  - $\deg(G,x)$: Degree of vertex $x \in V(G)$

Lazy simple random walk on (static) G: At time t, the walker moves from x to y w.p. $P(x,y)$ defined as follows:

$$P(x,y) := \begin{cases} \frac{1}{2\deg(G,x)} & (\text{if } \{x,y\} \in E(G)) \\ \frac{1}{2} & (\text{if } x = y) \\ 0 & (\text{otherwise}) \end{cases}.$$



For any connected graph G, let P be the transition matrix of the lazy simple RW on G. Then,

$$t_{\mathrm{hit}}(P) = O(\boldsymbol{n^3}).$$

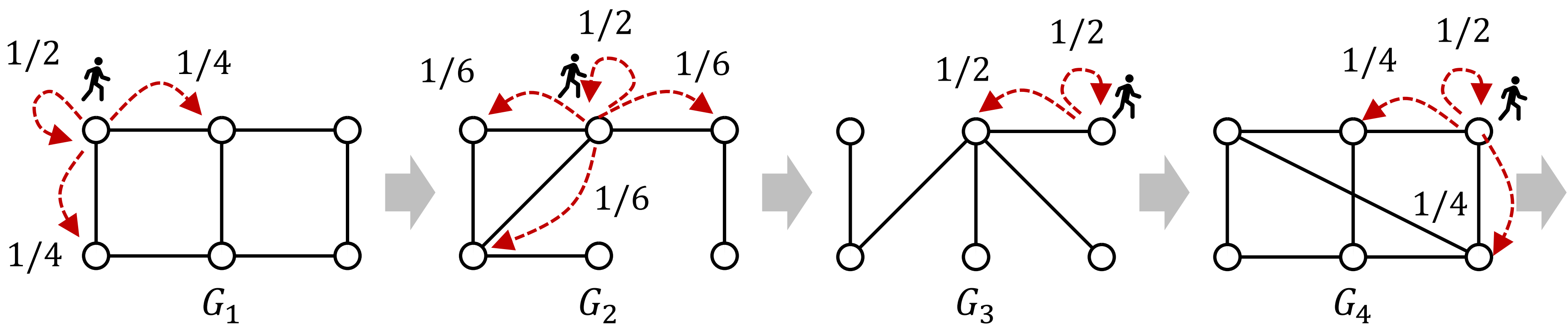
There exists a tight example (Lollipop graph)

[Aleliunas, Karp, Lipton, Lovász, Rackoff. FOCS 79]

✓ $G_1, G_2, \ldots$: Sequence of edge-changing n-vertex graphs

<u>*Lazy simple random walk* on $G_1, G_2, \ldots$</u> : At time t, the walker moves from x to y w.p. $P_t(x,y)$ defined as follows:

$$P_t(x,y) := \begin{cases} \frac{1}{2\deg(G_t,x)} & (\text{if } \{x,y\} \in E(G_t)) \\ \frac{1}{2} & (\text{if } x = y) \\ 0 & (\text{otherwise}) \end{cases}.$$



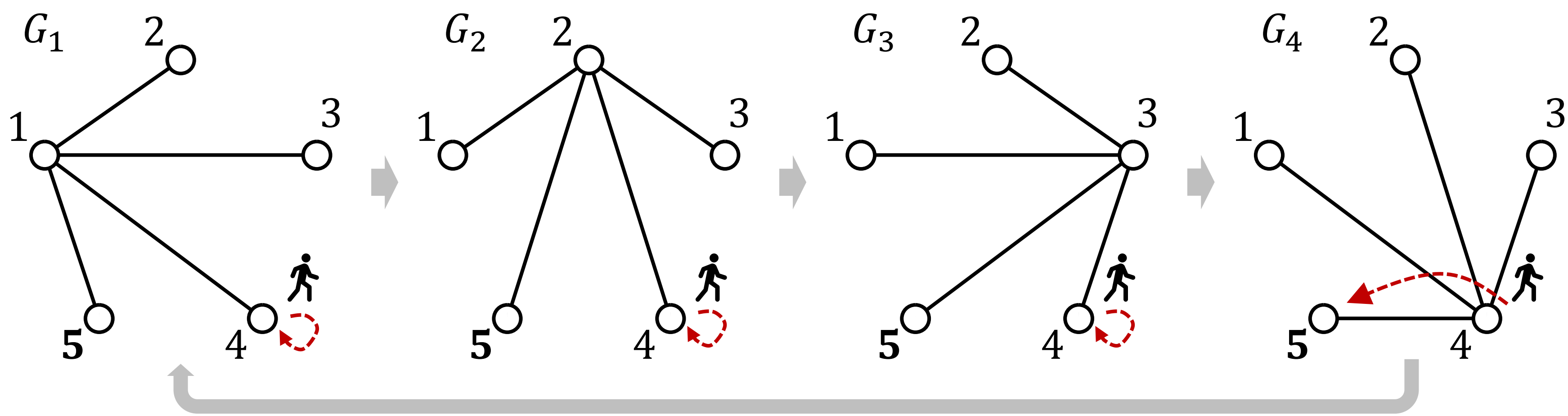
Ex. Lazy simple random walk on $G_1, G_2, \ldots$

Previous work: Exponential lower bound for LSRW

Sisyphus wheel. Sequence of star graphs $G_1, G_2, \ldots$ with $V(G_t) = \{1, \ldots, \boldsymbol{n}\}$, where the center changes periodically in $1, \ldots, \boldsymbol{n-1}$

For Sisyphus wheel $G_1, G_2, \ldots$, let P_t be the transition matrix of the *lazy simple random walk* on G_t. Then, $\boldsymbol{t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) = 2^{\Omega(n)}}$.

[Avin, Kouský, Lotler. ICALP 08, RS&A 18]



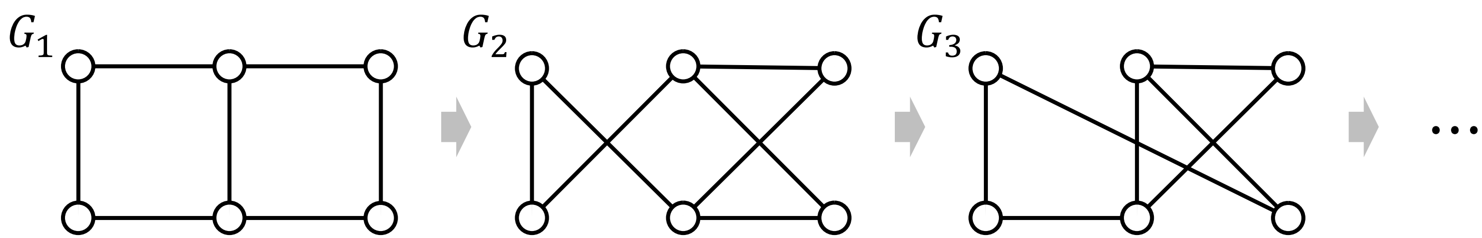
Ex of $n = 5$.

The walker must stay $n-2$ consecutive steps to reach the vertex n

∀sequence of connected graphs $G_1, G_2, \ldots$ with an invariant degree distribution, let P_t be the transition matrix of lazy simple walk on G_t. Then,

$$\boldsymbol{t}_{\mathbf{hit}}\left(\left(\boldsymbol{P}_t\right)_{t\geq 1}\right) = \boldsymbol{O}\left(\boldsymbol{n}^3 \log \boldsymbol{n}\right).$$

[Sauerwald, Zanetti. ICALP 19]



- ✓ In general, there exists a sequence of graphs s.t. HT is exponential (Sisyphus wheel)
- ✓ If the degree distribution is invariant, HT is polynomial

Application of Theorem 1: Lazy simple RW

We can apply Thm.1 for lazy simple RW on $G_1, G_2, \dots$ with time-invariant degree distribution!

- ✓ P_t (Transition matrix of LSRW on G_t) is irreducible, reversible and lazy (if G_t is connected)
- ✓ Stationary distribution of P_t is $\frac{\deg(G_t, x)}{2|E(G_t)|}$
  - ➢ Stationary distribution is invariant if degree dist. is !

Thm.1 (Hitting time). Suppose $(P_t)_{t\geq 1}$ satisfies the following:

- ✓ All $P_1, P_2, \dots$ are irreducible, reversible, and lazy
- ✓ **All $\boldsymbol{P_1, P_2}, \dots$ have the same stationary distribution $\boldsymbol{\pi}$**

Then, there is a constant $C > 0$ s.t. $\boldsymbol{t_{\mathrm{hit}}\big((P_t)_{t\geq 1}\big) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$

$$= C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), t_{\mathrm{hit}}(P_3), \dots\}$$

Corollary of Thm.1. $\forall$sequence of connected graphs $(G_t)_{t\geq 1}$ with an invariant degree distribution, let P_t be the transition matrix of the lazy simple RW on G_t. Then,

$$t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) = O\left(\boldsymbol{n^3}\right).$$

✓ Improves $O\left(n^3 \log n\right)$ bound of the previous work!

[Sauerwald, Zanetti. ICALP 19]

Remark.

✓ $t_{\mathrm{hit}}(P_1) = O\left(n^3\right), t_{\mathrm{hit}}(P_2) = O\left(n^3\right), \ldots$

HT of LSRW on G_1

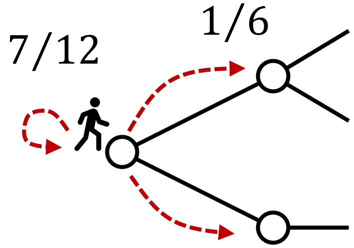
HT of LSRW on G_2

[Aleliunas et al. 79]
(previous work on static graphs)

Thm.1. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, lazy, and **have the same stationary distribution** $\boldsymbol{\pi}$. Then, $\boldsymbol{t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) \leq C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), \ldots\}}$.

Lazy Metropolis walk on (static) G. At time t, the walker moves from x to y w.p. $P(x,y)$ defined as follows:

$$P(x,y) := \begin{cases} \dfrac{1}{2\max\{\deg(G,x),\deg(G,y)\}} & \left(\text{if } \{x,y\} \in E(G)\right) \\ 1 - \displaystyle\sum_{w: w\sim x} P(x,w) & (\text{if } x = y) \\ 0 & (\text{otherwise}) \end{cases}$$



For any connected graph G, let P be the transition matrix of the lazy Metropolis walk on G. Then, $t_{\mathrm{hit}}(P) = O(\boldsymbol{n^2})$.

[Nonaka, Ono, Sadakane, Yamashita. Theoretical Compt. Sci. 10]

- ✓ Using local degree information achieves $O(n^2)$ hitting time
- ✓ There are no previous studies about dynamic cases

Cor. of Thm.1. ∀sequence of connected graphs $(G_t)_{t\geq 1}$, let P_t be the transition matrix of the lazy Metropolis W on G_t. Then,

$$t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) = O\left(\boldsymbol{n}^2\right).$$

Remark. Same bound as the static graph! i.e., LMW is **robust** for edge-changes

- ✓ Stationary distribution of LMW is the uniform distribution ($\pi(x) = 1/n$ for any graph)
  - ➢ Stationary distribution is invariant for any graphs!
- ✓ $t_{\mathrm{hit}}(P_1) = O\left(n^2\right), t_{\mathrm{hit}}(P_2) = O\left(n^2\right), \ldots$

HT of LMW on G_1 | HT of LMW on G_2

[Nonaka et al. 10]
(previous work on static graphs)

Thm.1. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, lazy, and **have the same stationary distribution** $\boldsymbol{\pi}$. Then, $t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) \leq C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), \ldots\}$.

Contents

✓ $\tau_y := \min\{t \geq 0 \mid X_t = y\}$: **Hitting time to $\boldsymbol{y}$** (random variable)

Remark. Hitting time $t_{\mathrm{hit}} = \max_{x,y \in V} \mathbf{E}_x[\tau_y]$ from definition

<u>**Hitting time lemma**</u>**.** Suppose $(P_t)_{t \geq 1}$ satisfies the following:

✓ All $P_1, P_2, \ldots$ are irreducible and reversible

✓ All $P_1, P_2, \ldots$ have the same stationary distribution π

Then, for any $w \in V$ and $T \geq 0$,

$$\Pr_{\pi}(\tau_w > T) \leq \left(1 - \frac{1}{\max_{t \geq 1} t_{\mathrm{hit}}(P_t)}\right)^T.$$

✓ For the walker <u>starting from the stationary distribution</u>, the hitting time to a vertex decreases exponentially

✓ $\tau_y := \min\{ t \geq 0 \mid X_t = y \}$: **Hitting time to $\boldsymbol{y}$** (random variable)

Hitting time lemma. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, and have the same stationary distribution π. Then, for any $w \in V$ and $T \geq 0$,

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✓ For the walker starting from the stationary distribution, the hitting time to a vertex decreases exponentially

➢ HTL implies that "Hitting time from stationary $\mathbf{E}_\pi[\tau_w]$" is bounded by $\max_{t \geq 1} t_{\mathrm{hit}}(P_t)$:

$$\mathbf{E}_\pi[\tau_w] \leq \sum_{T=0}^{\infty} \left(1 - \frac{1}{\max_{t \geq 1} t_{\mathrm{hit}}(P_t)}\right)^T = \max_{t \geq 1} t_{\mathrm{hit}}(P_t).$$

Proof of hitting time lemma (1/3)

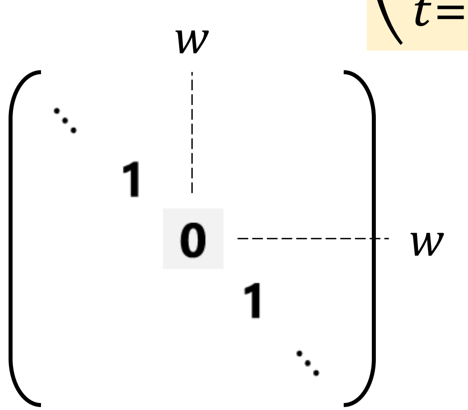
- ✓ $D_w \in \{0,1\}^{V \times V}$: diagonal matrix where $D_w(x,x) = \mathbf{1}_{x \neq w}$
- ➢ **Key observation.** τ_w can be expressed in terms of D_w:

$$\Pr_x(\tau_w > T, X_T = y) = \Pr_x\left(\bigwedge_{t=0}^{T} \{X_t \neq w\}, X_T = y\right)$$

$$= \left(\prod_{t=1}^{T} D_w P_t D_w\right)(x,y).$$

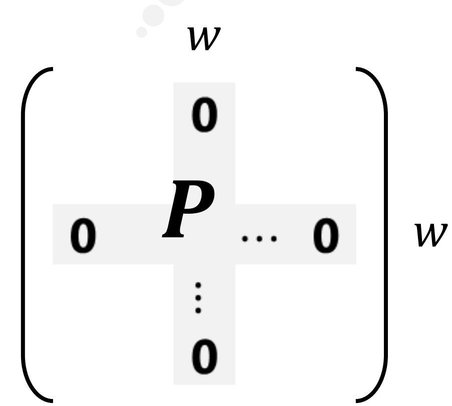
"Transitions that exclude reaching w"

$D_w =$



Identity matrix except that its (w,w)-entry is set to 0

$D_w P D_w =$



P except that its w-th row and column are set to 0

Proof of Hitting time lemma (2/3)

$$\Pr_{\pi}(\tau_w > T) = \sum_{x \in V} \pi(x) \sum_{y \in V} \left(\prod_{t=1}^{T} D_w P_t D_w \right)(x, y)$$

$$\leq \left\| \left(\prod_{t=1}^{T} D_w P_t D_w \right) \mathbf{1} \right\|_{2,\pi}$$

$$\leq \prod_{t=1}^{T} \rho(D_w P_t D_w)$$

$$\Pr_{x}(\tau_w > T, X_T = y) = \left(\prod_{t=1}^{T} D_w P_t D_w \right)(x, y)$$

Cauchy-Schwarz inequality

✓ $\|f\|_{2,\pi} \coloneqq \sqrt{\sum_{x \in V} \pi(x) f(x)^2}$ ($\ell_2(\pi)$-norm)

Courant-Fischer-Weyl Min-max theorem

✓ $\rho(A)$: the spectral radius of A

✓ The following lemma, a basic consequence of the *Perron-Frobenius theorem*, concludes the proof:

Lem. Suppose P is irreducible & reversible. Then, $\forall w \in V$, the spectral radius $\rho(D_w P D_w)$ of $D_w P D_w$ is bounded by $1 - \frac{1}{t_{\mathrm{hit}}(P)}$.

[Aldous, Fill 02]

$$\Pr_{\pi}(\tau_w > T) \le \prod_{t=1}^{T} \rho(D_w P_t D_w) \le \prod_{t=1}^{T} \left(1 - \frac{1}{t_{\mathrm{hit}}(P_t)}\right). \qquad \square$$

Hitting time lemma. Suppose all $P_1, P_2, \ldots$ are irreducible, and have the same stationary distribution π. Then, $\Pr_{\pi}(\tau_w > T) \le \left(1 - \frac{1}{\max_{t \ge 1} t_{\mathrm{hit}}(P_t)}\right)^T$.

Idea of proof for Theorem 1 (Hitting time)

Proof overview:

Hitting time lemma:

$$\mathbf{E}_{\pi}(\tau_w) \le \max_{t \ge 1} t_{\mathrm{hit}}(P_t) \, .$$

$t_{\mathrm{hit}}\big((P_t)_{t\ge1}\big) \le$ Time taken for a walker to converge π (from the worst initial pos.) $+ \max_{t\ge1} t_{\mathrm{hit}}(P_t)$

✓ Mixing time bound for time-**in**homogeneous Markov chain

✓ For time-homogeneous MC, the following is well-known:

Thm. Suppose P is irreducible, reversible and lazy. Then,

$$t_{\mathrm{mix}}^{(\infty)}(P) \le C t_{\mathrm{hit}}(P).$$

[Levin, Peres, Wilmer. 08]

Key tool 2: Mixing time ≲ Hitting time

Thm.2 (Mixing time). Suppose $(P_t)_{t\geq 1}$ satisfies the following:

- ✓ All $P_1, P_2, \dots$ are irreducible, reversible and lazy
- ✓ All $P_1, P_2, \dots$ have the same stationary distribution π

Then, there is a constant $C > 0$ s.t.

$$\boldsymbol{t}_{\mathbf{mix}}^{(\infty)}\left((\boldsymbol{P}_t)_{t\geq 1}\right) \leq \boldsymbol{C} \max_{t\geq 1} \boldsymbol{t}_{\mathbf{hit}}(\boldsymbol{P}_t).$$

($\ell_\infty(\pi)$-) **Mixing time** $\boldsymbol{t}_{\mathbf{mix}}^{(\infty)}\left((\boldsymbol{P}_t)_{t\geq 1}\right)$.

$$\boldsymbol{t}_{\mathbf{mix}}^{(\infty)}\left((\boldsymbol{P}_t)_{t\geq 1}\right)$$

$$:= \min\left\{t \geq 0\colon \max_{s\geq 0, x, y\in V} \left|\frac{(P_{s+1}P_{s+2}\cdots P_{s+t})(x,y)}{\pi(y)} - 1\right| \leq \frac{1}{2}\right\}$$

✓ The following bound of $t_{\mathrm{mix}}^{(\infty)}((P_t)_{t\geq 1})$ has been shown:

Suppose all $P_1, P_2, \ldots$ are irreducible, aperiodic, reversible, and have the same stationary distribution π. Then, ∃constant C s.t.

$$\boldsymbol{t_{\mathrm{mix}}^{(\infty)}((P_t)_{t\geq 1}) \leq C \max_{t\geq 1}\left(\frac{\log \pi_{\min}^{-1}}{1-\lambda_\star(P_t)}\right).}$$

[Saloff-Coste, Zúñiga. Stochastic Processes and their Applications 07]

$\lambda_\star(P)$: 2nd largest eigenvalue in absolute value of P

✓ For some cases (e.g., LSRW on expander graphs), this gives a better bound than our bound of $\max_{t\geq 1} t_{\mathrm{hit}}(P_t)$

✓ However, **there exists bad examples** (e.g, LSRW on cycles) where this bound gets $\max_{t\geq 1} t_{\mathrm{hit}}(P_t)\, \boldsymbol{\log n}$

For a probability vector $\mu \in [0,1]^V$, let

$$\Delta^{(\pi)}(\mu) := \left\| \frac{\mu}{\pi} - \mathbf{1} \right\|_{2,\pi}^2 = \sum_{x \in V} \pi(x) \left(\frac{\mu(x)}{\pi(x)} - 1 \right)^2.$$

Lem. $$\Delta^{(\pi)}(\mu P) \le \Delta^{(\pi)}(\mu) \left(1 - \frac{\Delta^{(\pi)}(\mu)}{t_{\mathrm{hit}}(P)} \right).$$

μP gets closer to π than μ in terms of $t_{\mathrm{hit}}(P)$

(dist. after one step)

Applying repeatedly

Lem. For $T \ge C \max_{t \ge 1} t_{\mathrm{hit}}(P_t)$, $\Delta^{(\pi)}(\mu P_1 \cdots P_T) \le \frac{1}{2}$.

$\ell_2(\pi)$-norm → $\ell_\infty(\pi)$-norm

Thm.2. For $T \ge C \max_{t \ge 1} t_{\mathrm{hit}}(P_t)$, $\max_{s \ge 0, x, y \in V} \left| \frac{(P_{s+1} \cdots P_{s+T})(x,y)}{\pi(y)} - 1 \right| \le \frac{1}{2}$.

Key technical lemma

Lem. [Mihail 89]

$$\Delta^{(\pi)}(\mu P) \le \Delta^{(\pi)}(\mu) - \boldsymbol{\mathcal{E}}_{\boldsymbol{P},\boldsymbol{\pi}}\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right).$$

Lem. $\boldsymbol{\mathcal{E}}_{\boldsymbol{P},\boldsymbol{\pi}}\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right) \ge \frac{\Delta^{(\pi)}(\mu)^2}{t_{\mathrm{hit}}(P)}.$

Lem.

$$\Delta^{(\pi)}(\mu P) \le \Delta^{(\pi)}(\mu)\left(1 - \frac{\Delta^{(\pi)}(\mu)}{t_{\mathrm{hit}}(P)}\right).$$

For $f \in \mathbb{R}^V$, let

$$\boldsymbol{\mathcal{E}}_{\boldsymbol{P},\boldsymbol{\pi}}(\boldsymbol{f}) \coloneqq \mathcal{E}_{P,\pi}(f,f) = \frac{1}{2}\sum_{x,y\in V} \pi(x)P(x,y)\big(f(x) - f(y)\big)^2$$

$$= \langle f, f\rangle_\pi - \langle f, Pf\rangle_\pi. \qquad \textit{(Dirichlet form)}$$

$\langle f, g\rangle_\pi \coloneqq \sum_{x\in V} \pi(x)f(x)g(x)$: π-inner product

Key technical lemma (Proof, 1/3)

Lem. For any irreducible & reversible P,

$$\mathcal{E}_{\boldsymbol{P},\boldsymbol{\pi}}\left(\frac{\boldsymbol{\mu}}{\boldsymbol{\pi}}\right) \geq \frac{\Delta^{(\pi)}(\mu)^2}{t_{\mathrm{hit}}(P)}.$$

- ✓ Let $g(x) := \left\|\frac{\mu}{\pi}\right\|_\infty - \frac{\mu(x)}{\pi(x)}$.
  - ➢ $\left\|\frac{\mu}{\pi}\right\|_\infty = \max_{x\in V}\frac{\mu(x)}{\pi(x)}$
- ✓ The proof consists of the following three statements:

1. $\mathcal{E}_{P,\pi}\left(\frac{\mu}{\pi}\right) = \langle g, g\rangle_\pi - \langle Pg, g\rangle_\pi$

2. $\langle Pg, g\rangle_\pi \leq \left(1 - \frac{1}{t_{\mathrm{hit}}(P)}\right)\langle g, g\rangle_\pi.$

3. $\langle g, g\rangle_\pi \geq \Delta^{(2,\pi)}(\mu)^2$

Key technical lemma (Proof, 2/3)

$$\mathcal{E}_{P,\pi}(f) := \frac{1}{2}\sum_{x,y\in V}\pi(x)P(x,y)\big(f(x)-f(y)\big)^2 = \langle f,f\rangle_\pi - \langle f,Pf\rangle_\pi$$

✓ $g(x) := \left\|\frac{\mu}{\pi}\right\|_\infty - \frac{\mu(x)}{\pi(x)}$.

1. Since $\frac{\mu(x)}{\pi(x)} - \frac{\mu(y)}{\pi(y)} = g(y) - g(x)$,

$$\mathcal{E}_{P,\pi}\left(\frac{\mu}{\pi}\right) = \mathcal{E}_{P,\pi}(g) = \langle g,g\rangle_\pi - \langle Pg,g\rangle_\pi.$$

2. Let $w \in V$ be a vertex s.t. $\frac{\mu(w)}{\pi(w)} = \left\|\frac{\mu}{\pi}\right\|_\infty$. Then, $g(w) = 0$ and

$$\langle Pg,g\rangle_\pi = \langle D_w P D_w g, g\rangle_\pi \le \rho(D_w P D_w)\langle g,g\rangle_\pi \le \left(1 - \frac{1}{t_{\mathrm{hit}}(P)}\right)\langle g,g\rangle_\pi.$$

D_w: Identity matrix except that its (w,w)-entry is 0

Lem. $\rho(D_w P D_w) \le 1 - \frac{1}{t_{\mathrm{hit}}(P)}$.

✓ $g(x) \coloneqq \left\|\frac{\mu}{\pi}\right\|_\infty - \frac{\mu(x)}{\pi(x)}.$

3. From

$$\Delta^{(\pi)}(\mu) = \sum_{x \in V} \pi(x) \left(\frac{\mu(x)}{\pi(x)} - 1\right)^2 = \sum_{x \in V} \pi(x) \left(\frac{\mu(x)}{\pi(x)}\right)^2 - 1 \le \left\|\frac{\mu}{\pi}\right\|_\infty - 1,$$

$$\langle g, g \rangle_\pi = \sum_{x \in V} \pi(x) \left(\left\|\frac{\mu}{\pi}\right\|_\infty - \frac{\mu(x)}{\pi(x)}\right)^2$$

$$= \left\|\frac{\mu}{\pi}\right\|_\infty^2 + \sum_{x \in V} \pi(x) \left(\frac{\mu(x)}{\pi(x)}\right)^2 - 2\left\|\frac{\mu}{\pi}\right\|_\infty$$

$$\ge \left(\left\|\frac{\mu}{\pi}\right\|_\infty - 1\right)^2 \ge \Delta^{(\pi)}(\mu)^2.$$

$$\sum_{x \in V} \pi(x) \left(\frac{\mu(x)}{\pi(x)}\right)^2 \ge 1$$

- ✓ We also studied other parameters of a random walk according to $(P_t)_{t\geq 1}$:
  - ➢ **Cover time**
  - ➢ Hitting and cover times of k-independent walkers
  - ➢ Coalescing time

Cover time $\boldsymbol{t}_{\mathbf{cov}} := \max_{x \in V} \mathbf{E}_x[\min\{\, t \geq 0 \mid \{X_0, X_1, \dots, X_t\} = V \,\}]$

✓ The expected # of steps for the walker to visit all vertices (from the worst initial position)

Write $\boldsymbol{t}_{\mathbf{cov}}(\boldsymbol{P})$ as the CT of the RW according to P

i.e., CT of a RW with a time-**in**variant transition matrix

✓ There is much previous work

Write $\boldsymbol{t}_{\mathbf{cov}}\big((\boldsymbol{P}_{\boldsymbol{t}})_{\boldsymbol{t}\geq\mathbf{1}}\big)$ as the CT of the RW according to $(P_t)_{t\geq 1}$

i.e., CT of a RW with time-varying transition matrices

✓ Not much is known

✓ There is much previous work for time-invariant P, e.g.,

For any connected graph G, let P be the transition matrix of the lazy simple RW on G. Then,

$$t_{\mathrm{cov}}(P) = O(\boldsymbol{n^3}).$$

There exists a tight example (Lollipop graph)

[Aleliunas, Karp, Lipton, Lovász, Rackoff. FOCS 79]

For any connected graph G, let P be the transition matrix of the lazy Metropolis walk on G. Then,

$$t_{\mathrm{cov}}(P) = O(\boldsymbol{n^2 \log n}).$$

There exists a tight example (glitter star)

[Nonaka, Ono, Sadakane, Yamashita. Theoretical Compt. Sci. 10]

For any irreducible P, $t_{\mathrm{cov}}(P) \leq \boldsymbol{t_{\mathrm{hit}}(P) \log n}$.

[Matthews. Annals of Proability 88]

Result (Cover time)

Thm.3 (Cover time). Suppose $(P_t)_{t\geq 1}$ satisfies the following:

✓ All $P_1, P_2, \ldots$ are irreducible, reversible, and lazy

✓ All $P_1, P_2, \ldots$ have the same stationary distribution π

Then, there is a constant $C > 0$ s.t.

$$\boldsymbol{t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t) \log n} .$$

✓ Multiplying $\boldsymbol{\max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$ by $\boldsymbol{O(\log n)}$ is sufficient to cover all vertices (even for the time-inhomogeneous case)

✓ Theorem 3 gives tight bounds for some cases

➢ Lazy Metropolis walk

Application of Theorem 3: Lazy Metropolis walk

Cor. of Thm.3. ∀sequence of connected graphs $(G_t)_{t\geq 1}$, let P_t be the transition matrix of the lazy Metropolis W on G_t. Then,

$$t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) = O\left(\boldsymbol{n^2 \log n}\right).$$

Remark. Same bound as the static graph! i.e., LMW is **robust** for edge-changes

✓ Stationary distribution of LMW is the uniform distribution ($\pi(x) = 1/n$ for any graph)

➢ Stationary distribution is invariant for any graphs!

✓ $t_{\mathrm{hit}}(P_1) = O\left(n^2\right), t_{\mathrm{hit}}(P_2) = O\left(n^2\right), \ldots$

HT of LMW on G_1

HT of LMW on G_2

[Nonaka et al. 10]
(previous work on static graphs)

Thm.3. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, lazy, and have the same stationary distribution π. Then, $\boldsymbol{t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t) \log n}$.

Proof sketch of Theorem 3

We can assume the initial position $\sim \pi$ from Thm. 2

Thm 2 (Mixing time).
$t_{\mathrm{mix}}^{(\infty)}\left((P_t)_{t\geq 1}\right) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)$.

✓ $\tau_{\mathrm{cov}} \coloneqq \min\{\, t \geq 0 \mid \{X_0, \dots, X_t\} = V \,\}$

✓ $\tau_y \coloneqq \min\{\, t \geq 0 \mid X_t = y \,\}$

For $T = \max_{t\geq 1} t_{\mathrm{hit}}(P_t) \log n$, Union bound + HTL implies

$$\Pr_\pi(\tau_{\mathrm{cov}} > Ti) = \Pr_\pi\left(\bigcup_{w\in V} \{\tau_w > Ti\}\right) \leq n\left(1 - \frac{1}{\max_{t\geq 1} t_{\mathrm{hit}}(P_t)}\right)^{Ti}$$

$$\leq n^{-(i-1)}.$$

Hitting time lemma.

$$\Pr_\pi(\tau_w > T) \leq \left(1 - \frac{1}{\max_{t\geq 1} t_{\mathrm{hit}}(P_t)}\right)^T.$$

Hence, $\mathbf{E}_\pi[\tau_{\mathrm{cov}}] = O(T)$.

Remark. Thm.3 is (possibly) weak for LSRW

Corollary of Thm.3. ∀sequence of connected graphs $(G_t)_{t\geq 1}$ with an invariant degree distribution, let P_t be the transition matrix of the lazy simple RW on G_t. Then,

$$t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) = O\left(\boldsymbol{n}^3 \log \boldsymbol{n}\right).$$

Q. Is it tight?

➢ Is there a bad sequence of graphs with an invariant degree dist. s.t. $t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) = \Omega\left(n^3 \log n\right)$?

There exists a tight example (Lollipop graph)

Time-invariant case:

For any connected graph G, let P be the transition matrix of the lazy simple RW on G. Then, $t_{\mathrm{cov}}(P) = O\left(\boldsymbol{n}^3\right)$. [Aleliunas, Karp, Lipton, Lovász, Rackoff. FOCS 79]

Thm.3. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, lazy, and have the same stationary distribution π. Then,

$$t_{\mathrm{cov}}\big((P_t)_{t\geq 1}\big) \leq C \max_{t\geq 1} \boldsymbol{t}_{\mathbf{hit}}(\boldsymbol{P_t}) \log n\,.$$

Conjecture. Suppose all $P_1, P_2, \ldots$ are irreducible, reversible, lazy, and have the same stationary distribution π. Then,

$$\boldsymbol{t}_{\mathbf{cov}}\big((\boldsymbol{P_t})_{\boldsymbol{t}\geq \mathbf{1}}\big) \leq \boldsymbol{C} \max_{\boldsymbol{t}\geq \mathbf{1}} \boldsymbol{t}_{\mathbf{cov}}(\boldsymbol{P_t}) \;\; (?)$$

- ✓ Is it true? Or a counter-example exists?
  - ➢ Do good tools like the Hitting time lemma exist?

    e.g., $\Pr_{\pi}(\tau_{\mathrm{cov}} > T) \leq \cdots$

Conclusion

✓ We give an upper bound on HT of a RW with <u>time-varying</u> transition matrices, in terms of HTs of <u>time-**in**variant</u> ones:

<u>Thm.1 (Hitting time).</u> Suppose $(P_t)_{t\geq 1}$ satisfies the following:

✓ All $P_1, P_2, \dots$ are irreducible, reversible, and lazy

✓ <u>**All $\boldsymbol{P_1, P_2, \dots}$ have the same stationary distribution $\boldsymbol{\pi}$**</u>

Then, there is a constant $C > 0$ s.t.

$$\boldsymbol{t_{\mathrm{hit}}\big((P_t)_{t\geq 1}\big) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$$
$$= C \max\{t_{\mathrm{hit}}(P_1), t_{\mathrm{hit}}(P_2), t_{\mathrm{hit}}(P_3), \dots\}.$$

✓ We also give upper bounds for the mixing and cover times in terms of $\boldsymbol{\max_{t\geq 1} t_{\mathrm{hit}}(P_t)}$

Summary: *Thank you for your attention!*

Lazy simple RW

Graph	t_{hit}	t_{cov}	
∀connected G (static)	$O(n^3)$		[Aleliunas et al. 79]
∃connected G (static)	$\Omega(n^3)$ (Lollipop)		[Feige. 95]
∃seq. of connected graphs $G_1, G_2, \dots$	$2^{\Omega(n)}$ (Sisyphus wheel)		[Avin, Kouský, Lotler. 08]
∀seq. of connected graphs $G_1, G_2, \dots$ with time-invariant degree dist.	$O(n^3 \log n)$	$O(n^3 \log^2 n)$	[Sauerwald, Zanneti. 19]
	$O(n^3)$	$O(n^3 \log n)$	[Shimizu, S. 23]

Lazy Metropolis walk

Graph	t_{hit}	t_{cov}	
∀connected G (static)	$O(n^2)$	$O(n^2 \log n)$	[Nonaka et al. 10]
∃connected G (static)	$\Omega(n^2)$ (e.g., path)	$\Omega(n^2 \log n)$ (glitter star)	
∀seq. of connected graphs $G_1, G_2, \dots$	$O(n^2)$	$O(n^2 \log n)$	[Shimizu, S. 23]

$$t_{\mathrm{hit}}\left((P_t)_{t\geq 1}\right) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t)$$

$$t_{\mathrm{cov}}\left((P_t)_{t\geq 1}\right) \leq C \max_{t\geq 1} t_{\mathrm{hit}}(P_t) \log n\,.$$