Uniqueness and Rapid Mixing in the Bipartite Hardcore Model

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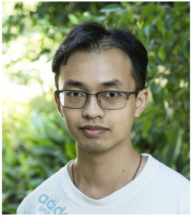
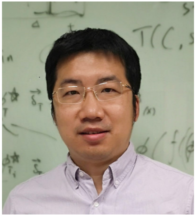
based on joint work with



Jingcheng Liu



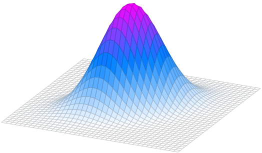
Yitong Yin

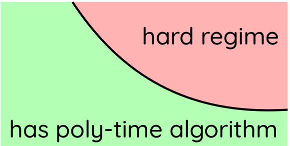
Sampling problem:

Draw (approximate) random samples from a distribution

Gibbs distribuiton:

- high-dimensional joint distribution
- described by few parameters and local interactions





Computational phase transition:

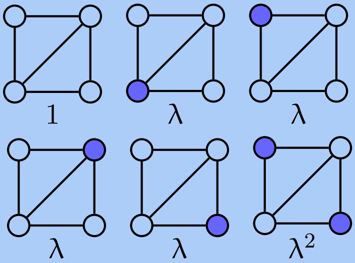
computational complexity of sampling problem changes sharply around certain parameter values

Hardcore model

- $G = ([n], E)$ with n vertices and max degree Δ.
- Fugacity $\lambda > 0$ is a real number.
- $\mathrm{Ind}(G) = \{S \subseteq [n] \mid S \text{ is an independent set}\}$.
- Gibbs distribution

$$\forall S \in \mathrm{Ind}(G), \quad \mu(S) := \frac{\lambda^{|S|}}{Z}, \quad \text{where } Z_G(\lambda) := \sum_{I \in \mathrm{Ind}(G)} \lambda^{|I|}.$$

an example



Partition function:

$$Z = 1 + 4\lambda + \lambda^2$$

This model is self-reducible

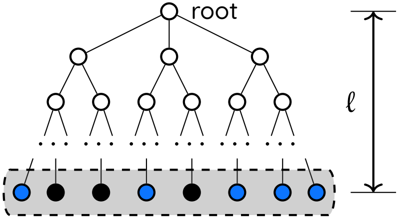
Computational phase transition

On Δ-regular tree:

uniqueness $\lambda_c(\Delta)$ non-uniqueness

$\lambda : 0$ ⟶ ∞

Uniqueness threshold: $\lambda_c(\Delta) := (\Delta - 1)^{(\Delta-1)}/(\Delta - 2)^{\Delta} \approx \frac{e}{\Delta}$



σ : boundary condition on level ℓ

Uniqueness Threshold

$\mathbf{Pr}_{S\sim\mu}[\text{root} \in S \mid \sigma]$ does not depend on σ when $\ell \to \infty$

if and only if $\lambda \le \lambda_c(\Delta)$

On general graph with maximum degree Δ:

easy $\lambda_c(\Delta)$ hard

$\lambda : 0$ ⟶ ∞

Computational phase transition:

- $\lambda < \lambda_c$: poly-time algorithm for approx. sampling [Wei06]
- $\lambda > \lambda_c$: no poly-time algorithm unless **NP = RP** [Sly10]

Computational phase transition

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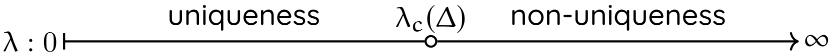

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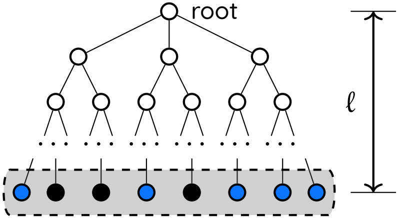

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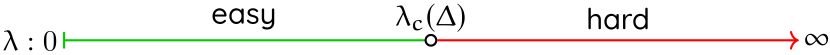

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Hardcore model on bipartite graph (weighted #BIS)

It is easy: there is a poly-time algorithm to find a matximum independent set in the bipartite graph (Kőnig's theorem[1]).

It is hard: many important problems are proved to be #BIS-equivalent or #BIS-hard under AP-reductions.

Selected examples

- stable matchings (counting)
- ferro. Potts model (parti. func.)
- ferro. Ising with mixed external fields (parti. func.)

[DGGJ04, GJ07, DGJ10, CGM12 DGJR12, GJ12a, BDG+13, LLZ14, GJ15, CGG+16, GŠVY16, GGY21,]

Conjecture[DGGJ04]:
#BIS represents an intermediate complexity class:

- it has no FPRAS in general
- it is easier than #SAT

[1] In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

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Previous algorithmic results

Low temperature regime (via polymer):

- α-expander bipartite graph:
  - $\lambda \geq (C_0\Delta)^{4/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [JKP20]
  - $\lambda \geq (C_1\Delta)^{6/\alpha}$, an $O(n \log n)$ time sampler [CGG+21]
  - $\lambda \geq (C_2\Delta)^{2/\alpha}$, an $n^{O(\log \Delta)}$ time sampler [FGKP23]
- Δ-regular α-expander bipartite graph:
  - $\lambda \geq \frac{f(\alpha)\log \Delta}{\Delta^{1/4}}$, an $n^{O(\Delta)}$ time sampler [JPP22]
- random Δ-regular bipartite graph:
  - $\Delta \geq \Delta_0, \lambda \geq \frac{\log^4 \Delta}{\Delta}$, an $n^{O(1)}$ time sampler [LLLM19]
  - $\Delta \geq \Delta_1, \lambda \geq \frac{50 \log^2 \Delta}{\Delta}$, an $n^{1+O(\frac{\log^2(\Delta)}{\Delta})}$ time sampler [JKP20]
  - $\Delta \geq \Delta_2, \lambda \geq \frac{100 \log \Delta}{\Delta}$, an $O(n \log n)$ time sampler [CGŠV22]
- unbalanced bipartite graph:
  - $6\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [CP20]
  - $3.4\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $n^{O(\log(\Delta_L\Delta_R))}$ time sampler [FGKP23]
  - $(1+e)\Delta_L\Delta_R\lambda \leq (1+\lambda)^{\frac{\delta_R}{\Delta_L}}$, an $O(n \log n)$ time sampler [BCP22]

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High temperature regime (via spatial mixing):

- general graph: if $\lambda < \lambda_c(\Delta)$, there is an $O(n \log n)$ time sampler
- bipartite graph: if $\lambda = 1, \Delta_L \leq 5$, an $O(n^2)$ time sampler [LL15]
  $(\lambda = 1 \wedge \lambda < \lambda_c(\Delta) \Leftrightarrow \Delta \leq 5)$

The low/high-temperature regime follows from the **weak spatial mixing**

σ_Λ : fixed configuration in Λ

Weak spatial mixing (WSM)

$\mathbf{Pr}[v \in S \mid \sigma_\Lambda]$ doesn't depend on σ_Λ as $\ell \longrightarrow +\infty$

high-temperature $\Leftrightarrow$ WSM

In bipartite graph, unlike general graph, we don't have a clear picture about when does the weak spatial mixing hold.

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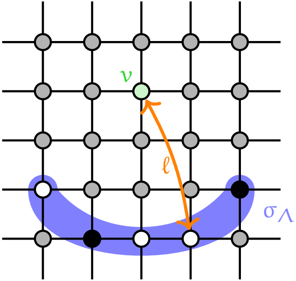

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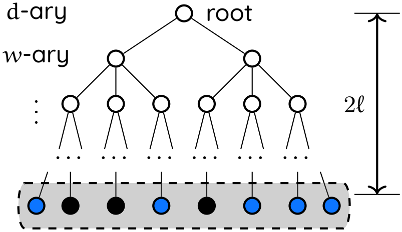

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Tree recursion of (d, w)-ary tree

$$F(x) := \lambda(1 + \lambda(1 + x)^{-w})^{-d}$$

diverge (> 1 fixed point)

converge ($= 1$ fixed point)

Definition

Let $\delta \in [0, 1)$ be a real number. The pair $(\lambda, d) \in \mathbb{R}^2_{>0}$ is δ-**unique** if for any $w \in \mathbb{R}_{>0}$, all fixpoints $\hat{x} = F(\hat{x})$ of F satisfy $F'(\hat{x}) \leq 1 - \delta$.

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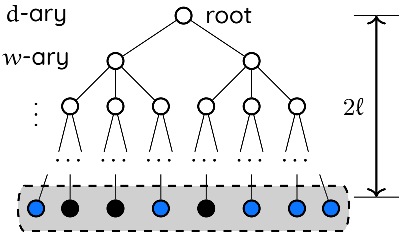

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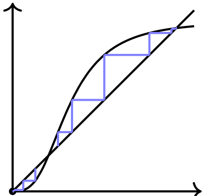

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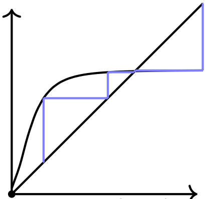

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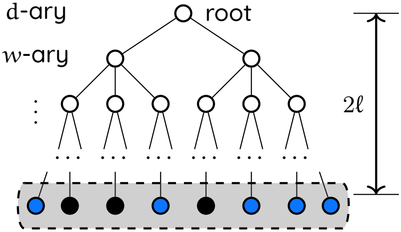

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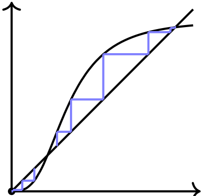

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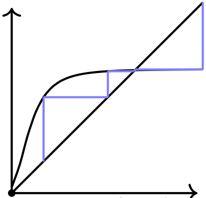

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Theorem

Fix any $\Delta = d + 1 \ge 3$ and any $\delta \in [0,1)$, the pair (λ, d) is $\frac{\delta}{10}$-unique if

$$\lambda \le (1-\delta)\lambda_c(\Delta) = (1-\delta)\frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}.$$

Theorem

For bipartite graph $G = (L \cup R, E)$ with maximum degree $\Delta_L = d + 1 \ge 2$ on L and fugacity $\lambda > 0$, let $n = |L|$, then for any $\delta \in (0,1)$, if (λ, d) is δ-unique, then we have a sampler for this hardcore model that runs in time

$$n\left(\frac{\Delta_L \log n}{\lambda}\right)^{O(C/\delta)}, \text{where } \begin{cases} C = O(1), & \Delta_L \ge 3 \\ C = (1+\lambda)^{10}, & \Delta_L = 2. \end{cases}$$

- When $\Delta_L = 1$, G is a forest.
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Glauber dynamics for Hardcore model:

start from an arbitrary independent set X_0;
for t from 1 to T **do**:

- pick a vertex $v \in V$ uniformly at random;
- with prob. $\frac{\lambda}{1+\lambda}$, let $S = X_{t-1} \cup \{v\}$;
  with prob. $\frac{1}{1+\lambda}$, let $S = X_{t-1} \setminus \{v\}$;
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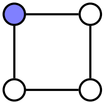

irreducible + aperiodic + reversible $\Longrightarrow X_t \sim \mu$ as $t \to \infty$

mixing time: essential running time of Glauber dynamics

$$T_{\mathrm{mix}} := \max_{X_0} \min\{t \mid D_{\mathrm{TV}}(X_t \parallel \mu) \leq 1/100\}$$

total variation distance: conanical distance between distributions

$$D_{\mathrm{TV}}(X_t \parallel \mu) := \frac{1}{2} \sum_{S \in \mathrm{Ind}(G)} |\mathbf{Pr}[X_t = S] - \mu(S)|$$

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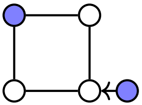

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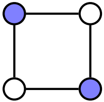

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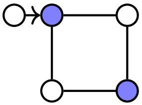

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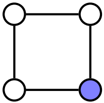

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Our results

Glauber dynamics for Hardcore model:

start from an arbitrary independent set X_0;
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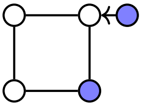

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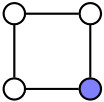

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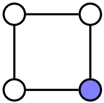

Theorem

For bipartite graph $G = (L \cup R, E)$ with maximum degree $\Delta_L = d + 1 \geq 3$ on L, $\delta \in (0, 1)$, and fugacity $\lambda \in (0, (1 - \delta)\lambda_c(\Delta))$. Then the mixing time of the Glauber dynamics is bounded as

$$T_{\text{mix}} \leq \left(\frac{\Delta \log n}{\lambda}\right)^{O(C/\delta)} \cdot n^3 \cdot \log \frac{1+\lambda}{\min\{1, \lambda\}}.$$

- When $\Delta_L \geq 3$, then $C = O(1)$.
- When $\Delta_L = 2$, (λ, d) is δ-unique, the bound holds with $C = (1 + \lambda)^{10}$.

Background

Proof outline

Fast sampler

Mixing of Glauber dynamics on $\mathbb{L} \cup \mathbb{R}$

Spectral independence

δ-uniqueness

Background

Let ν be a distribution over $\Omega = \{-1, +1\}^n$. $\forall \sigma \in \Omega, \quad \|\sigma\|_+ = |\{i \mid \sigma_i = 1\}|$

impose external field $\theta > 0$

$\theta * \nu$: a distribution on Ω:

$$\forall \sigma, \quad (\theta * \nu)(\sigma) \propto \nu(\sigma) \cdot \theta^{\|\sigma\|_+}$$

flip the distribution

$\overline{\nu}$: a distribution on Ω:

$$\forall \sigma, \quad \overline{\nu}(\sigma) = \nu(-\sigma)$$

- hardcore model: μ (fugacity λ) $\Longrightarrow$ $\theta * \mu$ (fugacity $\theta\lambda$)

For $0 < \theta \neq 1$, Field dynamics $P^{FD}_{\theta,\nu}$: Markov chain $(X_t)_{t \geq 0}$ on Ω:

X_0 is an arbitrary vector in Ω and let $s \in \{-1, +1\}$ so that $\theta^s < 1$;
for each $t > 0$:

1. **generate** $R \subseteq [n]$: **for** $i \in [n]$ with $X_{t-1}(i) = s$
   add i to R with prob. $1 - \theta^s$
2. **let** $X_t = \sigma$ with prob. $\mathbf{Pr}_{\sigma \sim \theta * \nu}[\sigma \mid \sigma_R = s]$

irreducible + aperiodic + reversible [CFYZ21] $\Longrightarrow$ $X_t \sim \nu$ as $t \to \infty$ 🤔

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Theorem ([CFYZ21, AJKPV22, CFYZ22, CE22])

Let $0 < \theta \neq 1$ and ν be a distribution over $\{-1, +1\}^n$ that

1. $\lambda * \nu$ is K-**marginally stable** for all λ between θ, 1,
2. $\lambda * \nu$ is η-**spectrally independent** for all λ between θ, 1,
3. the Glauber dynamics on $\theta * \nu$ mixes in time $\widetilde{O}(n)$,

then

$$1 \wedge 2 \Rightarrow T_{\text{mix}}(P^{\text{FD}}_{\theta,\nu}) \approx \max\{\theta, 1/\theta\}^{\eta \cdot \text{poly}(K)}.$$

$$1 \wedge 2 \wedge 3 \Rightarrow \text{sampler for } \nu \text{ in time } \widetilde{O}(n) \cdot \max\{\theta, 1/\theta\}^{\eta \cdot \text{poly}(K)}$$

$$1 \wedge 2 \wedge 3 \overset{\text{Var}}{\Rightarrow} T_{\text{mix}}(P^{\text{GD}}_{\nu}) \approx \widetilde{O}(n) \cdot \underbrace{n \cdot \max\{\theta, 1/\theta\}^{\eta \cdot \text{poly}(K)}}_{\text{relaxation time}}$$

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Let ν be a distribution over $\{-1,+1\}^n$ and $X \sim \nu$ be a random vector.

influence matrix $\Psi_\nu \in \mathbb{R}^{n\times n}$

$$\Psi_\nu(i,j) := \begin{cases} 0, & \text{if } \mathbf{Pr}_\nu[i] \in \{0,1\} \\ \mathbf{Pr}_\nu[j \mid i] - \mathbf{Pr}_\nu\left[j \mid \bar{i}\right] \end{cases}$$

$$i = \{X_i = +1\}, \bar{i} = \{X_i = -1\}$$

$\mathrm{Corr}(X) \in \mathbb{R}^{n\times n}$

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- Ψ_ν is similar to $\mathrm{Corr}(X)$

η-spectral independence (in ∞-norm)

$\forall \Lambda \subseteq [n]$ with $|\Lambda| \le n-2$, and $\forall \tau \in \Omega(\nu_\Lambda)$, $\|\Psi_{\nu^\tau}\|_\infty \le \eta$

K-marginal stability

there is $\rho \in \{\nu, \bar{\nu}\}$ that for $i \in [n]$, $S \subseteq \Lambda \subseteq [n] \setminus \{i\}$, $\tau \in \Omega(\rho_\Lambda)$,

$$R_i^\tau \le K \cdot R_i^{\tau_S} \text{ and } \rho_i^\tau(-1) \ge K^{-1}$$

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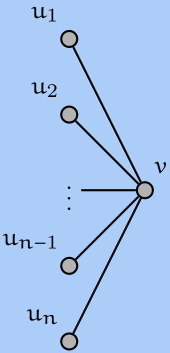
Background

Proof outline

- Fast sampler
- Mixing of Glauber dynamics on $L \cup R$
- Spectral independence
- δ-uniqueness

Proof outline

example



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μ: Gibbs distribution of the hardcore model

$\|\Psi_\mu\|_\infty$ is unbounded

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What could we do? 🤔

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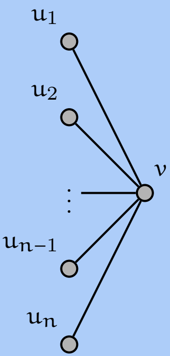

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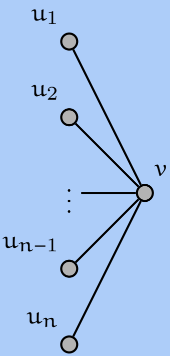

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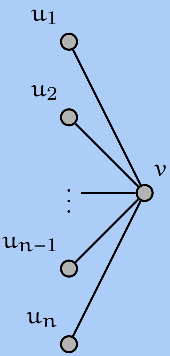

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$$\nu \xrightarrow[\substack{\text{BHC}(\lambda,\lambda) \qquad P^{FD}_{\theta,\nu} \text{ with } \theta = \Theta(\frac{\Delta \log n}{\lambda}) > 1}]{O(1/\delta)\text{-spectrally independent} \qquad O(1)\text{-marginally stable}} \theta * \nu$$

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Glauber dynamics mixes in $\widetilde{O}(n)$

- fast sampler for ν in time $n \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ ($\Rightarrow$ fast sampler for μ)
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For $\nu = \mu_L$ on BHC(λ, α): δ-uniqueness $\Longrightarrow$ $O(1/\delta)$-spectral independence

$\Delta_L = d + 1$

$F(x) = \lambda(1 + \alpha(1 + x)^{-w})^{-d}$

$\forall w > 0, \forall \hat{x}, F'(\hat{x}) \leq 1$

uniqueniss regime

Fix $d \geq 1$, the pair (λ, d, α) is unique if the point (λ, α) is on above of the following parametric curve for $w > d^{-1}$:

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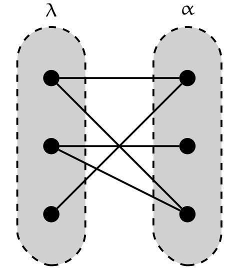
BHC(λ, λ) — $P^{FD}_{\theta,\nu}$ with $\theta = \Theta(\frac{\Delta \log n}{\lambda}) > 1$ — BHC$(\theta\lambda, \lambda)$

Glauber dynamics mixes in $\widetilde{O}(n)$✅

- fast sampler for ν in time $n \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ ($\Rightarrow$ fast sampler for μ)
- Glauber dynamics on ν mixes in time $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$

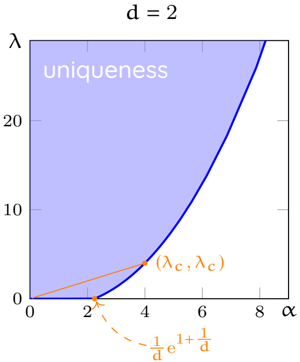
For $\nu = \mu_L$ on BHC(λ, α): δ-uniqueness $\Longrightarrow$ $O(1/\delta)$-spectral independence



$\Delta_L = d + 1$

$F(x) = \lambda(1 + \alpha(1 + x)^{-w})^{-d}$

$\forall w > 0, \forall \hat{x}, F'(\hat{x}) \le 1$



uniqueniss regime

Fix $d \ge 1$, the pair (λ, d, α) is unique if the point (λ, α) is on above of the following parametric curve for $w > d^{-1}$:

$$\begin{cases} \alpha(w) = \frac{d^w (w+1)^{w+1}}{(dw-1)^{w+1}} \\ \lambda(w) = \frac{w^d (d+1)^{d+1}}{(dw-1)^{d+1}} \end{cases}$$

Background

Proof outline

Fast sampler

Mixing of Glauber dynamics on $L \cup R$

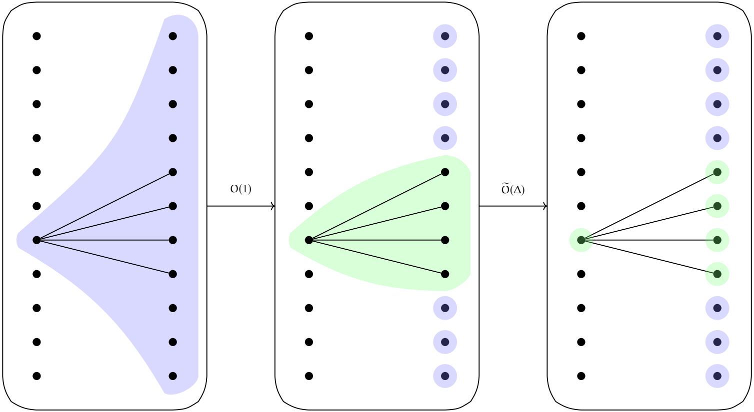
Spectral independence

δ-uniqueness

Proof outline: mixing of GD on μ

Glauber dynamics on ν mixes in time $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$

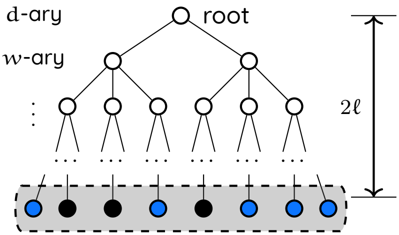
To get an algorithm that only updates each site seperately:



- This algorithm also runs in $n^2 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$ round
- A vertex $u \in R$ is updated with rate 1 in each round
- The Glauber dynamics on μ mixes in time $n^3 \cdot (\frac{\Delta \log n}{\lambda})^{O(1/\delta)}$

Could be implemented by block factorization [CMT15, CP20, CLV21].

Proof outline: spectral independence



- $\nu = \mu_L$ for BHC(λ, α) that is δ-unique
- reduce the general case to the $(d+1, w+1)$-regular tree via the SAW tree [CLV21] and a special potential function [LL15]

$$\Phi(x) := \log(\log(1+x))$$

$$\Phi'(x) =: \phi(x) = \frac{1}{(1+x)\log(1+x)}$$

- recursion on (d, w)-ary tree

$$F(x) = \lambda(1 + \alpha(1+x)^{-w})^{-d}$$

the total influence is bounded by

$$\sum_{\ell=1}^{+\infty}\left(\sum_{v \in L_T(2\ell)} |\Psi_\mu(\text{root}, v)|\right) \overset{[CLV21]}{\leq} O(1) \cdot \sum_{\ell=1}^{+\infty}\left\{\sup_R (\Phi \circ F \circ \Phi^{-1})'(\Phi(R))\right\}^\ell$$

$$= O(1) \cdot \sum_{\ell=1}^{+\infty}\left\{\sup_R \frac{\phi(F(R))}{\phi(R)} F'(R)\right\}^\ell$$

$$\leq O(1) \cdot \sum_{\ell=1}^{+\infty}(1-\delta)^\ell = O(1/\delta).$$

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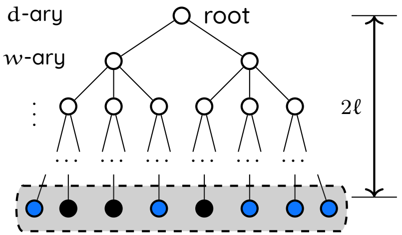

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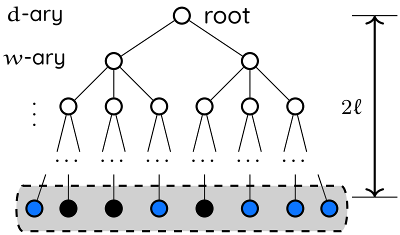

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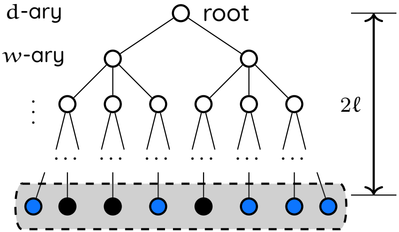

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Proof outline: contraction ($\delta = 0$)

δ-uniqueness

The tuple (λ, d, α) is δ-unique if $\forall w \in \mathbb{R}_{>0}$, and $\forall \hat{x}$, $F'(\hat{x}) \le 1 - \delta$.

contraction

If (λ, d, α) is δ-unique, $d \ge 1$, $\sup_{x \ge 0} H(x) := \sup_{x \ge 0} \frac{\phi(F(x))}{\phi(x)} F'(x) \le 1 - \delta$.

1. w could be eliminated by a change of variable: $z = 1 + \alpha(1+x)^{-w}$ [LL15],

$$\sup_{x \ge 0} H(x) = \sup_{z \in [1, 1+\alpha]} U(\lambda, d, \alpha; z)$$

$$H(x) = H(\lambda, d, \alpha, w; x)$$

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- $\partial_\lambda U \le 0$ and $\partial_\alpha U \ge 0$

2. There are function $c_1(x) > 0, c_2(x) > 0$ (when $x > 0$) that
$$H'(x) = c_1(x) \cdot (1 - H(x)) + c_2(x) \cdot (\alpha d - (x+1)^w (F(x) + 1))$$

3. Uniqueness regime

uniqueniss boundary

parametric curve:

$$\begin{cases} \alpha(w) = \frac{d^w (w+1)^{w+1}}{(dw-1)^{w+1}} \\ \lambda(w) = \frac{w^d (d+1)^{d+1}}{(dw-1)^{d+1}} \end{cases}$$

On the boundary (α, λ):
$\exists$ unique $w = w_c$:

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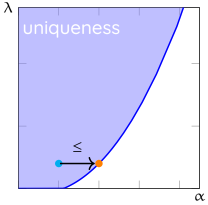

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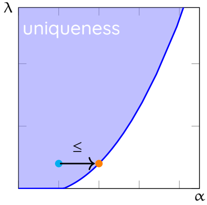
3. Uniqueness regime



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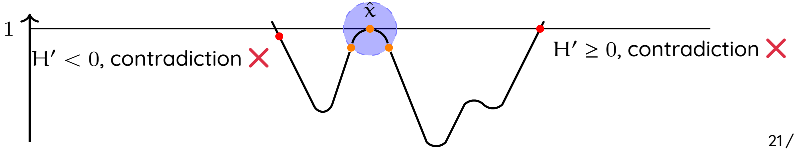
where $\hat{x}$ is the unique fixpoint.

4. Move (λ, α) to the boundary and let $w = w_c$ in $H(x) = \frac{\phi(F(x))}{\phi(x)} F'(x)$.

$$H(\hat{x}) = 1, H'(\hat{x}) = 0, H''(\hat{x}) < 0$$



Proof outline: contraction ($\delta > 0$)

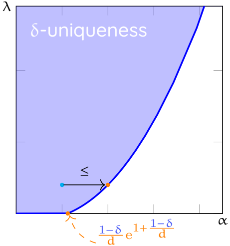
For simplicity, we assume $\delta = 0$. The $\delta > 0$ case could be handled by a similar high level idea.

1. w could be eliminated $\cdots$
2. There are function $c_1(x) > 0$, $c_2(x) > 0$ (when $x > 0$) that

$$H'(x) = c_1(x) \cdot ((1-\delta) - H(x)) + c_2(x) \cdot B_\delta(x), \text{ where}$$

$$B_\delta(x) = w \log(x+1) \left(\alpha d \cdot \frac{x+1}{F(x)+1} - (x+1)^{w+1} \right) + \delta(x+1)(\alpha + (x+1)^w)$$

3. Uniqueness regime



No parametric equation avaliable

On the boundary (α, λ): $\exists$ unique $w = w_c$:

$$\begin{cases} (1-\delta)(\hat{x}+1)(\alpha + (1+\hat{x})^w) - \alpha d w \hat{x} = 0 \\ w \log(1+\hat{x})(\alpha d - (1+\hat{x})^{w+1}) + \delta(\hat{x}+1)(\alpha + (\hat{x}+1)^w) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F'(\hat{x}) = 1-\delta \\ w \log(1+\hat{x})(\alpha d - (1+\hat{x})^{w+1}) + \delta(\hat{x}+1)(\alpha + (\hat{x}+1)^w) = 0 \end{cases},$$

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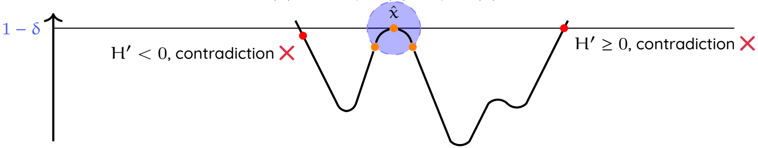

Background

Proof outline

- Fast sampler
- Mixing of Glauber dynamics on $L \cup R$
- Spectral independence
- δ-uniqueness

Proof outline: δ-uniqueness

$F(x) = \lambda(1+\alpha(1+x)^{-w})^{-d}$

δ-uniqueness

The tuple (λ, d, α) is δ-unique if $\forall w \in \mathbb{R}_{>0}$, and $\forall \hat{x}$, $F'(\hat{x}) \leq 1-\delta$.

- The requiremnt on fixpoint is not easy to use 🤔.
- $(\hat{x}, d, \alpha, w)$ determines a unique λ: $\lambda(\hat{x}) = \hat{x}(1+\alpha(1+\hat{x})^{-w})^{d}$
- Change the coordinates: $(\lambda, d, \alpha, w) \leftrightarrow (\hat{x}, d, \alpha, w)$
- $F'(\hat{x}) \leq 1-\delta \Leftrightarrow$ $\underbrace{(1-\delta)(\hat{x}+1)(\alpha+(1+\hat{x})^{w}) - \alpha d w \hat{x}}_{=:T_\delta(\hat{x})} \geq 0$

has more than one fixed points

$\cup$ $F'(\hat{x}) > 1-\delta$ at some fixed point $\hat{x}$

Fix d, α, w, a typical case is

- as $w \to +\infty$, we have $\lambda^{\delta}_{i} \to 0$
- $\lambda \geq \lambda^{\delta}_{2,c}$ implies δ-uniqueness
- fix d, α, critical $\hat{x}, w$ arise when $\begin{cases} T_\delta(x) = 0 \\ \partial_w \lambda^{\delta}_{2}(w) = 0 \end{cases} \overset{\delta=0}{\Longrightarrow}$ parametric curve

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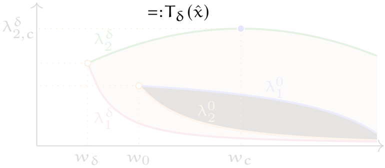

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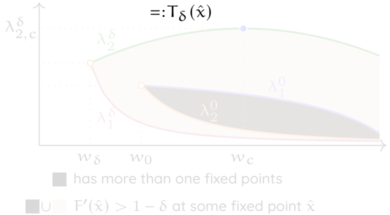

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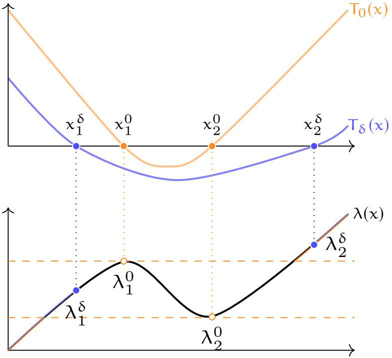

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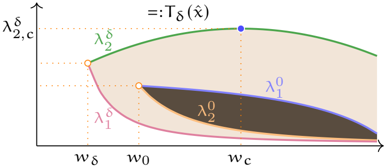

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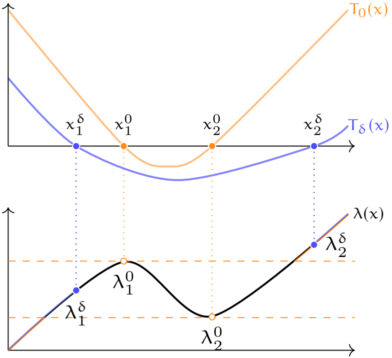


- as $w \to +\infty$, we have $\lambda^\delta_i \to 0$
- $\lambda \ge \lambda^\delta_{2,c}$ implies δ-uniqueness
- fix d, α, critical $\hat{x}, w$ arise when $\begin{cases} T_\delta(x) = 0 \\ \partial_w \lambda^\delta_2(w) = 0 \end{cases} \overset{\delta=0}{\Longrightarrow}$ parametric curve

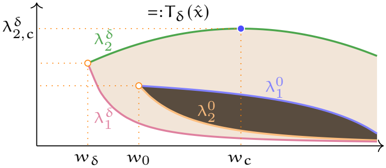
Proof outline: δ-uniqueness

$F(x) = \lambda(1+\alpha(1+x)^{-w})^{-d}$

δ-uniqueness

The tuple (λ, d, α) is δ-unique if $\forall w \in \mathbb{R}_{>0}$, and $\forall \hat{x}$, $F'(\hat{x}) \le 1-\delta$.

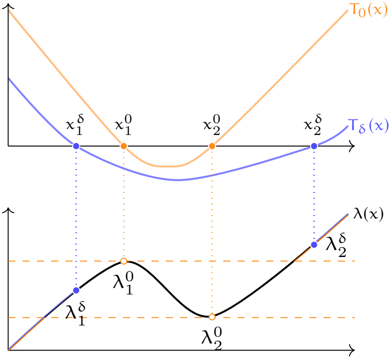
- The requiremnt on fixpoint is not easy to use 🤔.
- $(\hat{x}, d, \alpha, w)$ determines a unique λ: $\lambda(\hat{x}) = \hat{x}(1+\alpha(1+\hat{x})^{-w})^d$
- Change the coordinates: $(\lambda, d, \alpha, w) \leftrightarrow (\hat{x}, d, \alpha, w)$
- $F'(\hat{x}) \le 1-\delta \Leftrightarrow$ $\underbrace{(1-\delta)(\hat{x}+1)(\alpha+(1+\hat{x})^w) - \alpha d w \hat{x}}_{=:T_\delta(\hat{x})} \ge 0$



■ has more than one fixed points

■∪■ $F'(\hat{x}) > 1-\delta$ at some fixed point $\hat{x}$

Fix d, α, w, a typical case is



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Open problems

- Uniqueness regime for negative λ.
  May lead to better regime for k-CNF via LLL.
- Remove the depedency on degree Δ in the running time.
- Currently, we are able to show that the Glauber dynamics on μ mixes in time $\widetilde{O}(n^3)$ via a comparison argument.
  We want to know whether it actually mixes in $O(n \log n)$ time.

Thank you

arXiv:2305.00186

Take home message: considering a bipartite hardcore model with **different fugacities on both side** might result in unexpected flexibility in the analysis