LOCAL-TO-GLOBAL CONTRACTION IN SIMPLICIAL COMPLEXES

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Abstract. We give a local-to-global principle for relative entropy contraction in simplicial complexes. This is similar to the local-to-global principle for variances obtained by Alev and Lau (2020).

1. Introduction

High-dimensional expanders are a powerful new tool for analyzing mixing times of Markov chains which has resulted in many recent successes (Anari et al., 2019; Cryan et al., 2021; Anari et al., 2020; Chen et al., 2020c; Feng et al., 2020; Chen et al., 2020a,b). The main insight is to recast Markov chains as random walks over faces of simplicial complexes of the same cardinality. The mixing time of these "global" random walks can be bounded via analyzing certain "local" walks, thanks to the inductive structure of simplicial complexes. These local walks are defined over the skeleton of the faces of the simplicial complex, and are typically much easier to analyze. (Detailed definitions are given in Section 2.)

This "local-to-global" principle is the key to all of the recent development along this line of research. Roughly speaking, the local walks contain enough information about the high-dimensional global walks, yet in the mean time they are much more tractable to analyze. This methodology turns out to be very powerful (Dinur and Kaufman, 2017; Oppenheim, 2018; Kaufman and Oppenheim, 2020; Kaufman and Mass, 2020; Liu et al., 2020). In particular, the result of Kaufman and Oppenheim (2020) plays a crucial role in resolving the long-standing open problem for the expansion of the basis-exchange graphs for matroids by Anari et al. (2019). However, these early results do not give non-trivial bounds unless the eigenvalues of the local walks are sufficiently small. In contrast, Alev and Lau (2020) obtain a great improvement where the bound is always meaningful as long as the local walks are not completely trivial, which is currently the best result regarding the local-to-global principle for eigenvalues (or equivalently for variance contractions).

Despite these progresses, in many cases, there are inherent losses if one bounds the mixing times via eigenvalues or variance contractions (Levin and Peres, 2017). A tighter relationship can be obtained via relative entropy contractions (Diaconis and Saloff-Coste, 1996). Although relative entropy decay is typically much harder to analyze than eigenvalues, this method has been successfully applied to many problems, e.g. (Jerrum et al., 2004; Morris, 2013). In the high-dimensional expander context, Cryan et al. (2021) obtained optimal relative entropy contraction for the aforementioned matroid setting to get sharp mixing time bounds.

In this note we give a local-to-global principle for relative entropy contraction (Theorem 9). Our result is similar to that of Alev and Lau (2020) for variance contraction, in the sense that our bounds are always non-trivial as long as the local walks are not completely trivial. In fact, our proof also works for variance contraction. However, in that case our bounds are quantitatively no better than that of Alev and Lau (2020), because the local spectral gaps should satisfy the "trickling-down" theorem of Oppenheim (2018) (see Section 3.1 for some detailed comparison). In some interesting special cases, our bound actually coincides with the counterpart of Alev and Lau (2020) (see Corollary 5). In any case, our main interest lies in the local-to-global principle for relative entropy contraction, which would lead to improved mixing time bounds in certain applications.

Our main result, Theorem 9, is independently obtained by Chen et al. (2020b, Theorem 5.4). In fact, this theorem is one of the key ingredients of Chen et al. (2020b) to obtain optimal mixing times for Glauber dynamics over spin systems up to certain uniqueness conditions. We refer the interested

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readers to their work for applications. In this note we focus on the local-to-global principle for relative entropy contraction.

2. Preliminaries

The main things that need to be defined in this section are simplicial complexes, as well as the distributions and random walks over them.

2.1. Simplicial Complexes.

An (abstract) simplicial complex $\mathcal{C} = (E, \mathcal{S})$ is a tuple of a ground set of elements E, and a nonempty downwards closed collection of sets $\mathcal{S}$ (faces):

- $\emptyset \in \mathcal{S}$;
- if $S \in \mathcal{S}$, $T \subseteq S$, then $T \in \mathcal{S}$.

A pure simplicial complex has maximal sets of the same cardinality d. We denote by $\mathcal{C}(k)$ the collection of sets of size k, where $0 \leqslant k \leqslant d$.

We define the following distributions over $\mathcal{S}$. First, a distribution π_d is given on the top level, $\mathcal{C}(d)$, and then lower level distributions defined as follows:

$$\pi_k(S) \propto \sum_{T \in \mathcal{C}(d) : T \supset S} \pi_d(T), \qquad 0 \leqslant k < d.$$

Let $D_k := \operatorname{diag}(\pi_k)$. For a face $S \in \mathcal{S}$, the link $\mathcal{C}_S = (E \setminus S, \mathcal{S}_S)$ is also a simplicial complex, where $\mathcal{S}_S = \{T \mid T \subseteq E \setminus S, T \cup S \in \mathcal{S}\}$. We may similarly define distributions $\pi_{S,k}$ over $\mathcal{C}_S(k)$ simply as

$$\pi_{S,k}(T) \propto \pi_{|S|+k}(T \cup S), \qquad 0 \leqslant k \leqslant d - |S|.$$

It is also convenient to equip $\mathcal{C}$ with a weight function w, recursively defined as follows:

$$w(S) := \begin{cases} \pi(S) & \text{if } |S| = d, \\ \sum_{T \supset S,\, |T|=|S|+1} w(T) & \text{if } |S| < d. \end{cases}$$

We will refer to the weight function of $\mathcal{C}(k)$ as w_k, and, as before, the weight function of $\mathcal{C}_S(k)$ will be denoted by $w_{S,k}$.

2.2. Up and down random walks.

There are two natural exchange walks on $\mathcal{C}(k)$, $P_k^{\wedge}$ and $P_k^{\vee}$, which start by adding or removing an element and coming back to $\mathcal{C}(k)$. We call these walks "global" as they are defined over the whole of $\mathcal{C}(k)$. They are comprised by the same two parts:

- "Going-up", $P_k^{\uparrow}$; starting from a set $S \in \mathcal{C}(k)$, we add an element $i \in E \setminus S$ with probability $\propto \pi_{k+1}(S \cup i)$.
- "Going-down", $P_k^{\downarrow}$; starting from a set $S \in \mathcal{C}(k)$, we remove an element $i \in S$ uniformly at random.

We can now write

$$P_k^{\wedge} = P_k^{\uparrow} P_{k+1}^{\downarrow}, \qquad P_k^{\vee} = P_k^{\downarrow} P_{k-1}^{\uparrow}. \tag{1}$$

We shall also use the notation $P_{S,k}^{\wedge}$ and $P_{S,k}^{\vee}$, where $S \in \mathcal{S}$, to denote the walks on $\mathcal{C}_S(k)$. It is easy to check the detailed balance condition to see that $P_{S,k}^{\wedge}$ and $P_{S,k}^{\vee}$ are reversible with respect to $\pi_{S,k}$.

The "local" walks that will be particularly interesting are the walks $P_{S,1}^{\wedge}$ and $P_{S,2}^{\vee}$ for every face $S \in \mathcal{S}$ with $|S| \leqslant d - 2$. These walks are not completely local as they have weights that count higher level supersets. One can observe that the transition matrices of these walks are cospectral, because they are of the form AB and BA.

The walks $G_S := 2P_{S,1}^{\wedge} - I$ are the non-lazy version of the local walks $P_{S,1}^{\wedge}$. A simplicial complex $\mathcal{C}$ is called a $(a_0, \ldots, a_{d-2})$-local-spectral expander if for any $S \in \mathcal{C}(k)$, where $0 \leqslant k \leqslant d - 2$,

$$\lambda_2(G_S) \leqslant a_k,$$

where $\lambda_2(\cdot)$ is the second largest eigenvalue. We call the vector $(a_0, \ldots, a_{d-2})$ satisfying the above a *spectral profile* of $\mathcal{C}$.

Theorem 1 (Main Theorem of Alev and Lau, 2020). *Let $\mathcal{C}$ be a simplicial complex that is a $(a_0, \ldots, a_{d-2})$-local-spectral expander. Then, for any $2 \leqslant k \leqslant d$,*

$$\lambda_2(P_k^{\vee}) \leqslant 1 - \frac{1}{k}\prod_{i=0}^{k-2}(1-a_i)\,.$$

This is an example of a "local-to-global" theorem which is the type of theorem we are aiming for.
We define the *Dirichlet form* of a reversible Markov chain P, over state space Ω, as

$$\mathcal{E}_P\,(f,g) := \sum_{x,y\in\Omega} \pi(x)f(x)\big[\mathbf{I}-P\big](x,y)g(y) = f^{\mathsf{T}}\mathrm{diag}(\pi)(I-P)g,$$

where f, g are two functions over Ω, and π is the stationary distribution of P. Let the variance of f be

$$\mathrm{Var}_{\pi}\,(f) := \mathbb{E}_{\pi}\,f^2 - (\mathbb{E}_{\pi}\,f)^2\,.$$

The Poincaré inequality for P is a variational way of characterizing the spectral gap:

$$1-\lambda_2(P) = \inf\left\{\frac{\mathcal{E}_P\,(f,f)}{\mathrm{Var}_{\pi}\,(f)} \;\middle|\; f:\Omega\to\mathbb{R}\;,\;\mathrm{Var}_{\pi}\,(f)\neq 0\right\}. \tag{2}$$

For $k \geqslant 2$ and a function $f^{(k)} : \mathcal{C}(k) \to \mathbb{R}$, define $f^{(i)} : \mathcal{C}(i) \to \mathbb{R}$ for $1 \leqslant i \leqslant k-1$ such that

$$f^{(i)} := \prod_{j=i}^{k-1} P_j^{\uparrow} f^{(k)}. \tag{3}$$

For variances, we have

$$\mathcal{E}_{P_k^{\vee}}\left(f^{(k)}, f^{(k)}\right) = \mathrm{Var}_{\pi_k}\left(f^{(k)}\right) - \mathrm{Var}_{\pi_{k-1}}\left(f^{(k-1)}\right); \tag{4}$$

$$\mathcal{E}_{P_{k-1}^{\wedge}}\left(f^{(k-1)}, f^{(k-1)}\right) = \mathrm{Var}_{\pi_{k-1}}\left(f^{(k-1)}\right) - \mathrm{Var}_{\pi_k}\left(P_k^{\downarrow} f^{(k-1)}\right). \tag{5}$$

These equalities imply that the Poincaré inequalities for the walks $P_k^{\vee}$ and $P_{k-1}^{\wedge}$ are equivalent to variance contraction for the "half" walks $P_k^{\downarrow}$ and $P_{k-1}^{\uparrow}$, respectively.

Moreover, let the (normalised) relative entropy of $f : \Omega \to \mathbb{R}_{\geqslant 0}$ be

$$\mathrm{Ent}_{\pi}\,(f) := \mathbb{E}_{\pi}(f\log f) - \mathbb{E}_{\pi}\,f\log\mathbb{E}_{\pi}\,f,$$

where we follow the convention that $0\log 0 = 0$. A related notion is the Kullback–Leibler (KL) divergence $D(\tau \parallel \pi) := \sum_{x\in\Omega}\tau(x)\log\left(\frac{\tau(x)}{\pi(x)}\right)$, where τ and π are two distributions over the same Ω. Indeed, $D(\tau \parallel \pi) = \mathrm{Ent}_{\pi}\,(f)$ where $f = \frac{\tau}{\pi}$. The modified log-Sobolev constant (Bobkov and Tetali, 2006) is defined as

$$\rho(P) := \inf\left\{\frac{\mathcal{E}_P\,(f,\log f)}{\mathrm{Ent}_{\pi}\,(f)} \;\middle|\; f:\Omega\to\mathbb{R}_{\geqslant 0}\;,\;\mathrm{Ent}_{\pi}\,(f)\neq 0\right\}. \tag{6}$$

A related quantity is the relative entropy contraction ratio.

For entropies and $f^{(k)} : \mathcal{C}(k) \to \mathbb{R}_{\geqslant 0}$, we get inequalities instead of equalities as is the case for variances, (4) and (5),

$$\mathcal{E}_{P_k^{\vee}}\left(f^{(k)}, \log f^{(k)}\right) \geqslant \mathrm{Ent}_{\pi_k}\left(f^{(k)}\right) - \mathrm{Ent}_{\pi_{k-1}}\left(f^{(k-1)}\right); \tag{7}$$

$$\mathcal{E}_{P_{k-1}^{\wedge}}\left(f^{(k-1)}, \log f^{(k-1)}\right) \geqslant \mathrm{Ent}_{\pi_{k-1}}\left(f^{(k-1)}\right) - \mathrm{Ent}_{\pi_k}\left(P_k^{\downarrow} f^{(k-1)}\right). \tag{8}$$

Thus, entropy contractions of $P_k^{\downarrow}$ and $P_{k-1}^{\uparrow}$ imply modified log-Sobolev inequalities of $P_k^{\vee}$ and $P_{k-1}^{\wedge}$, respectively, but not the other way around, unlike the variance case.

One important result regarding the spectral profile of a simplicial complex is the trickling down theorem due to Oppenheim (2018). Here we give an alternative variational proof.

Theorem 2 (Trickling down theorem, Oppenheim, 2018). *Let $\mathcal{C}$ be a simplicial complex that is a γ-local-spectral expander at level k, meaning that $\lambda_2(G_T) \leqslant \gamma < 1$ for all $T \in \mathcal{C}(k)$. Then, if $\lambda_2(G_S) < 1$ (connectedness) for some $S \in \mathcal{C}(k-1)$,*

$$\lambda_2(G_S) \leqslant \frac{\gamma}{1-\gamma}.$$

Proof. It suffices to prove the theorem for $k = 1$. For any face S, let $D_{S,1} := \mathrm{diag}(\pi_{S,1})$, and note the decomposition

$$D_1 = \sum_{v \in \mathcal{C}(1)} \pi_1(v) \overline{D_{v,1}}, \tag{9}$$

where the overline denotes the extension by zeros to the appropriate dimension. Also, for any face S, $G_S = \mathrm{diag}(w_{S,1})^{-1} W_S$, where $W_S(u,v) := w_{S,2}(\{u,v\})$. We also have the following decomposition,

$$G_\emptyset = \mathrm{diag}(w_1)^{-1} W_\emptyset = \frac{1}{w(\emptyset)} D_1^{-1} \sum_{v \in \mathcal{C}(1)} \overline{W_v} = D_1^{-1} \sum_{v \in \mathcal{C}(1)} \frac{\overline{\mathrm{diag}(w_{v,1}) G_v}}{w(\emptyset)},$$

or equivalently,

$$G_\emptyset = D_1^{-1} \sum_{v \in \mathcal{C}(1)} \pi_1(v) \overline{D_{v,1} G_v}. \tag{10}$$

Moreover, notice that $G_\emptyset(v,u) = \pi_{v,1}(u)$ and for any function $f : \mathcal{C}(1) \to \mathbb{R}$,

$$G_\emptyset f(v) = \sum_{u \in \mathcal{C}(1)} G_\emptyset(v,u) f(u) = \overline{\pi_{v,1}} f. \tag{11}$$

Given these, for any function $f : \mathcal{C}(1) \to \mathbb{R}$, we can write

$$\begin{aligned}
\mathcal{E}_{G_\emptyset}(f,f) &= f^T D_1 (I - G_\emptyset) f \\
\text{(by (9) and (10))} \quad &= \sum_{v \in \mathcal{C}(1)} \pi_1(v) \left[f^T \overline{D_{v,1}} \left(I - \overline{G_v} \right) f \right] \\
\text{(where } f_v \text{ is } f \text{ restricted to } \mathcal{C}_v(1)) \quad &= \sum_{v \in \mathcal{C}(1)} \pi_1(v) \mathcal{E}_{G_v}(f_v, f_v) \\
\text{(because } \lambda_2(G_v) \leqslant \gamma) \quad &\geqslant (1-\gamma) \sum_{v \in \mathcal{C}(1)} \pi_1(v) \mathrm{Var}_{\pi_{v,1}}(f_v) \\
&= (1-\gamma) \sum_{v \in \mathcal{C}(1)} \pi_1(v) \left[f^T \overline{D_{v,1}} f - (\overline{\pi_{v,1}} f)^2 \right] \\
\text{(by (9) and (11))} \quad &= (1-\gamma) \left[\mathrm{Var}_{\pi_1}(f) + (\pi_1 f)^2 - \sum_{v \in \mathcal{C}(1)} \pi_1(v) \left([G_\emptyset f](v) \right)^2 \right] \\
&= (1-\gamma) \left[\mathrm{Var}_{\pi_1}(f) - \mathrm{Var}_{\pi_1}(G_\emptyset f) \right].
\end{aligned}$$

Since the inequality above holds for any f, we can choose $f = v_i$, where v_i is an eigenvector of $G_\emptyset$ corresponding to eigenvalue λ_i. Then,

$$\mathcal{E}_{G_\emptyset}(v_i, v_i) \geqslant (1-\gamma) \left[\mathrm{Var}_{\pi_1}(v_i) - \mathrm{Var}_{\pi_1}(G_\emptyset v_i) \right],$$

which simplifies into

$$(1-\lambda_i) v_i^T D_1 v_i \geqslant (1-\gamma)(1-\lambda_i^2) v_i^T D_1 v_i.$$

Thus, $(1-\lambda_i) \geqslant (1-\gamma)(1-\lambda_i^2)$. In particular, if $\lambda_2(G_\emptyset) < 1$, then

$$\lambda_2(G_\emptyset) \leqslant \frac{\gamma}{1-\gamma}.$$

□

An interesting question is to find a similar theorem for entropies or modified log-Sobolev constants. However, a straightforward generalisation of the proof above would involve the ratio between $\mathrm{Ent}_{\pi_1}(G_{\emptyset}f)$ and $\mathrm{Ent}_{\pi_1}(f)$. For variances, the ratio between $\mathrm{Var}_{\pi_1}(G_{\emptyset}f)$ and $\mathrm{Var}_{\pi_1}(f)$ can be related to λ_2, as shown in the proof. For relative entropy, that ratio does not seem to be directly related to $\rho(G_{\emptyset})$. In fact, ρ is the change rate of the relative entropy for the continuous-time Markov chain (Bobkov and Tetali, 2006), and can be related to the change rate for the discrete-time chain if there is no negative eigenvalue, as shown by Miclo (1997). However, this is not the case for $G_{\emptyset}$. It remains open to find relative entropy contraction descent similar to Theorem 2.

3. Decay of variance

We first study the local-to-global principle for variances. We show a theorem similar to the main result of Alev and Lau (2020). The bound we obtain will be given by a recursion, and seems to be incomparable to that of Alev and Lau (2020) at first sight. However, it turns out that our bound is weaker for spectral profiles satisfying the trickling down theorem, and it coincides with the bound of Alev and Lau (2020) when the trickling down theorem is tight. Nonetheless, our proof approach is different from that of Alev and Lau (2020), and our approach has the advantage that it generalises to the entropy case as shown in Section 4.

We use the following decomposition, which was shown by Cryan et al. (2021). For completeness we give a proof here.

Lemma 3. *Let $k \geqslant 2$ and $f^{(k)} : \mathcal{C}(k) \to \mathbb{R}$ be a function on $\mathcal{C}(k)$. Then,*

$$\mathrm{Var}_{\pi_k}\left(f^{(k)}\right) = \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S)\mathrm{Var}_{\pi_{S,2}}\left(f_S^{(2)}\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right),$$

where $f_S^{(2)}(T) := f^{(k)}(S \cup T)$ for $T \in \mathcal{C}_S(2)$. Moreover, the same decomposition holds for $f^{(k)} : \mathcal{C}(k) \to \mathbb{R}_{\geqslant 0}$ with $\mathrm{Var}\,()$ replaced by $\mathrm{Ent}\,()$.

Proof. We may assume that $\mathbb{E}_{\pi_k} f^{(k)} = 0$. By direct calculation, for any $I \in \mathcal{C}(k)$,

$$\pi_k(I) = \sum_{S \in \mathcal{C}(k-2), S \subset I} \pi_{k-2}(S)\pi_{S,2}(I \setminus S).$$

Notice that for any $S \in \mathcal{C}(k-2)$, $\mathbb{E}_{\pi_{k-2}} f^{(k-2)} = \mathbb{E}_{\pi_k} f^{(k)} = 0$, and

$$f^{(k-2)}(S) = \sum_{T \in \mathcal{C}_S(2)} \frac{2w(S \cup T)}{w(S)} f^{(k)}(S \cup T) = \mathbb{E}_{\pi_{S,2}} f_S^{(2)}.$$

Thus we have

$$\begin{aligned}\mathrm{Var}_{\pi_k}\left(f^{(k)}\right) &= \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S)\mathrm{Var}_{\pi_{S,2}}\left(f_S^{(2)}\right) + \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S)\left(\mathbb{E}_{\pi_{S,2}} f_S^{(2)}\right)^2 \\ &= \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S)\mathrm{Var}_{\pi_{S,2}}\left(f_S^{(2)}\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right).\end{aligned}$$

The proof for $\mathrm{Ent}\,()$ is completely analogous. □

Now we are ready to show the local-to-global principle for variances.

Theorem 4. *Let $\mathcal{C}$ be a simplicial complex that is a $(a_0, \dots, a_{d-2})$-local-spectral expander, and let $s_k := \frac{2}{1+a_k}$. Then, for any $2 \leqslant k \leqslant d$,*

$$\lambda_2(P_k^{\vee}) = \lambda_2(P_{k-1}^{\wedge}) \leqslant \frac{1}{\nu_{k-2}},$$

where ν_k is recursively defined as

(12) $$\nu_k = s_k - \frac{s_k - 1}{\nu_{k-1}}, \qquad \nu_0 = s_0.$$

Proof. From the assumption on local expansion we get that for any $S \in \mathcal{C}(k-2)$, where $2 \leqslant k \leqslant d$,

$$\lambda_2(G_S) \leqslant a_{k-2} \implies \lambda_2(P^{\vee}_{S,2}) \leqslant \frac{1+a_{k-2}}{2}.$$

Then, the spectral gap $\lambda(P^{\vee}_{S,2}) = 1 - \lambda_2(P^{\vee}_{S,2}) \geqslant \frac{1-a_{k-2}}{2}$. By (2), this implies that for any $g : \mathcal{C}_S(2) \to \mathbb{R}$,

$$\mathcal{E}_{P^{\vee}_{S,2}}(g, g) \geqslant \left(\frac{1-a_{k-2}}{2}\right) \mathrm{Var}_{\pi_{S,2}}(g).$$

Remembering that $s_{k-2} = \frac{2}{1+a_{k-2}}$, and (4), we obtain

$$\mathrm{Var}_{\pi_{S,2}}(g) \geqslant s_{k-2} \mathrm{Var}_{\pi_{S,1}}\left(P^{\uparrow}_{S,1} g\right). \tag{13}$$

We finish the proof by an induction on k. The base case of $k = 2$ is straightforward by noticing that $\nu_0 = s_0$. For the induction step on $k \geqslant 3$, by Lemma 3, we have

$$\begin{aligned}
\mathrm{Var}_{\pi_k}\left(f^{(k)}\right) &= \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,2}}\left(f^{(2)}_S\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right) \\
\text{(by (13))} \quad &\geqslant s_{k-2} \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,1}}\left(P^{\uparrow}_{S,1} f^{(2)}_S\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right) \\
&= (1-\epsilon) s_{k-2} \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,1}}\left(P^{\uparrow}_{S,1} f^{(2)}_S\right) \\
&+ \epsilon s_{k-2} \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,1}}\left(P^{\uparrow}_{S,1} f^{(2)}_S\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right), \qquad (14)
\end{aligned}$$

where $\epsilon \geqslant 0$ will be chosen later. A similar decomposition holds for level $k-1$,

$$\mathrm{Var}_{\pi_{k-1}}\left(f^{(k-1)}\right) = \sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,1}}\left(f^{(1)}_S\right) + \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right),$$

where $f^{(1)}_S(e) := f^{(k-1)}(S \cup \{e\})$ for $e \in \mathcal{C}_S(1)$. Together with the induction hypothesis this implies that

$$\sum_{S \in \mathcal{C}(k-2)} \pi_{k-2}(S) \mathrm{Var}_{\pi_{S,1}}\left(f^{(1)}_S\right) \geqslant (\nu_{k-3} - 1) \mathrm{Var}_{\pi_{k-2}}\left(f^{(k-2)}\right). \tag{15}$$

Finally, notice that $P^{\uparrow}_{S,1} f^{(2)}_S = f^{(1)}_S$. Plugging (15) into (14) and choosing $\epsilon = \frac{s_{k-2}-1}{\nu_{k-3} s_{k-2}}$,

$$\begin{aligned}
\mathrm{Var}_{\pi_k}\left(f^{(k)}\right) &\geqslant (1-\epsilon) s_{k-2} \mathrm{Var}_{\pi_{k-1}}\left(f^{(k-1)}\right) \\
&= \left(s_{k-2} - \frac{s_{k-2}-1}{\nu_{k-3}}\right) \mathrm{Var}_{\pi_2}\left(f^{(k-1)}\right) = \nu_{k-2} \mathrm{Var}_{\pi_{k-1}}\left(f^{(k-1)}\right).
\end{aligned}$$

By (2) and (4), this translates to the spectral gap bound $1 - \lambda_2(P^{\vee}_k) \geqslant 1 - \frac{1}{\nu_{k-2}}$, namely, $\lambda_2(P^{\vee}_k) \leqslant \frac{1}{\nu_{k-2}}$. ☐

A particular interesting case is when $a_{k-2} = \frac{\gamma}{1-(d-k)\gamma}$, for any $\gamma \leqslant 1/(d-1)$. This profile of a's is exactly the result of repeatedly applying the trickling down theorem, starting from level $d-2$ with $a_{d-2} = \gamma$, down to level 0.

Corollary 5. *Let $\mathcal{C}$ be a pure simplicial complex of dimension d and suppose that $a_{d-2} \leqslant \gamma \leqslant 1/(d-1)$. Then, for any $2 \leqslant k \leqslant d$,*

$$\lambda_2(P^{\vee}_k) = \lambda_2(P^{\wedge}_{k-1}) \leqslant 1 - \frac{1}{k}\left(\frac{1-(d-1)\gamma}{1-(d-k)\gamma}\right).$$

Proof. By the trickling down theorem (Theorem 2), we immediately get that $a_{k-2} \leqslant \frac{\gamma}{1-(d-k)\gamma}$. Due to Theorem 4, it suffices to prove that $\frac{1}{\nu_{k-2}} \leqslant 1 - \frac{1}{k}\left(\frac{1-(d-1)\gamma}{1-(d-k)\gamma}\right)$. For $k = 2$, the inequality holds trivially by the definition of ν_0. Suppose that the inequality holds up to dimension $k - 1 \geqslant 2$. Then,

$$\frac{1}{\nu_{k-3}} \leqslant 1 - \frac{1}{k-1}\left(\frac{1-(d-1)\gamma}{1-(d-k+1)\gamma}\right),$$

and as $a_{k-2} \leqslant \frac{\gamma}{1-(d-k)\gamma}$,

$$s_{k-2} = \frac{2}{a_{k-2}+1} \geqslant \frac{2(1-(d-k)\gamma)}{1-(d-k-1)\gamma}.$$

As in Theorem 4,

$$\begin{aligned}
\nu_{k-2} &= s_{k-2} - \frac{s_{k-2}-1}{\nu_{k-3}} \\
&\geqslant s_{k-2} - (s_{k-2}-1)\left[1 - \frac{1}{k-1}\left(\frac{1-(d-1)\gamma}{1-(d-k+1)\gamma}\right)\right] \\
&= 1 + (s_{k-2}-1)\frac{1}{k-1}\left(\frac{1-(d-1)\gamma}{1-(d-k+1)\gamma}\right) \\
&\geqslant 1 + \left(\frac{1-(d-k+1)\gamma}{1-(d-k-1)\gamma}\right)\frac{1}{k-1}\left(\frac{1-(d-1)\gamma}{1-(d-k+1)\gamma}\right) \\
&= 1 + \frac{1}{k-1}\left(\frac{1-(d-1)\gamma}{1-(d-k-1)\gamma}\right),
\end{aligned}$$

the reciprocal of which is

$$\frac{1}{\nu_{k-2}} \leqslant 1 - \frac{1}{k}\left(\frac{1-(d-1)\gamma}{1-(d-k)\gamma}\right).$$

The corollary follows by induction together with Theorem 4. □

In particular, by setting $\gamma = 1/d$, this retrieves Alev and Lau (2020, Corollary 1.6).

Corollary 6 (Alev and Lau, 2020, Corollary 1.6)**.** *If $\mathcal{C}$ is a $\frac{1}{d}$-local-spectral expander at level $d-2$, i.e. $a_{d-2} \leqslant 1/d$, then for any $2 \leqslant k \leqslant d$,*

$$\lambda_2(P_k^\vee) \leqslant 1 - \frac{1}{k^2}.$$

Corollary 5 and Corollary 6 are alternative derivations of a useful result of Alev and Lau (2020), which requires minimal assumptions on the expansion of the local graphs nearest to the top level. If we are studying the uniform distribution on the top level, these graphs are unweighted, as opposed to the lower level graphs that have weights that count some difficult to analyze and possibly intractable quantities.

An important advantage of our method is that it can also be used when we assume entropy contraction factors on the local walks. However, we do not know a counterpart to the trickling down theorem for entropies.

The contraction ratio ν_k is given by the recursion in (12). Next we will solve the recursion to give an explicit expression. Do the following substitution $x_k := \frac{\nu_k}{\nu_k - 1}$ and notice $x_0 = \frac{\nu_0}{\nu_0-1} = 1 + \frac{1+a_0}{1-a_0}$. The recurrence simplifies into

$$x_k = \frac{1+a_k}{1-a_k} \cdot x_{k-1} + 1, \tag{16}$$

and can be solved $x_k = \sum_{i=0}^k S_i^k + 1$, where $S_i^k := \prod_{j=i}^k \frac{1+a_j}{1-a_j} = \prod_{j=i}^k \frac{1}{s_j - 1}$. Then,

$$\nu_k = \frac{x_k}{x_k - 1} = 1 + \frac{1}{\sum_{i=0}^k S_i^k}. \tag{17}$$

Our second largest eigenvalue bound is

$$\gamma_k = \frac{1}{\nu_{k-2}} = 1 - \frac{1}{\sum_{i=0}^{k-2} S_i^{k-2} + 1}. \tag{18}$$

3.1. **Comparison with Alev and Lau (2020).** Given a spectral profile $(a_0, \ldots, a_{d-2})$, recall our second largest eigenvalue bound (18). In contrast, Theorem 1 (Alev and Lau, 2020, Theorem 1.5) achieves a different upper bound

$$\gamma_{k,AL} := 1 - \frac{1}{k} \prod_{i=0}^{k-2} (1 - a_i). \tag{19}$$

We call a spectral profile *admissible* if

(1) for all $0 \leqslant i \leqslant d-2$, $a_i < 1$;
(2) for all $1 \leqslant i \leqslant d-2$, $a_{i-1} \leqslant \frac{a_i}{1-a_i}$.

Note that the first condition here ensures that the random walk over the links are all connected, and the second condition ensures that the spectral profile is consistent with the trickling down theorem, Theorem 2.

Our bound in (18) is no better than the bound of (19) by Alev and Lau (2020) when $(a_0, \ldots, a_{d-2})$ is admissible.

Proposition 7. *Let $(a_0, \ldots, a_{d-2})$ be an admissible spectral profile. For any $0 \leqslant k \leqslant d-2$,*

$$\gamma_k \geqslant \gamma_{k,AL}.$$

To show Proposition 7, we need to first show a lemma.

Lemma 8. *Let $(a_0, \ldots, a_{d-2})$ be an admissible spectral profile. For any $1 \leqslant k \leqslant d-2$,*

$$a_k(k+1) + \prod_{i=0}^{k} (1 - a_i) \geqslant 1.$$

Proof. We do an induction on k. For $k = 1$, we have

$$2a_1 + (1 - a_0)(1 - a_1) = 1 + (a_1 + a_0 a_1 - a_0) \geqslant 1,$$

where the inequality is due to $a_0 \leqslant \frac{a_1}{1-a_1}$. For the induction step, suppose that the lemma holds for some $k \geqslant 1$, namely, $\prod_{i=0}^{k}(1 - a_i) \geqslant 1 - a_k(k+1)$. Thus,

$$\begin{aligned} a_{k+1}(k+2) + \prod_{i=0}^{k+1}(1 - a_i) &\geqslant a_{k+1}(k+2) + (1 - a_{k+1})(1 - a_k(k+1)) \\ &= 1 + (k+1)(a_{k+1} + a_k a_{k+1} - a_k) \geqslant 1, \end{aligned}$$

where the last inequality is because $a_k \leqslant \frac{a_{k+1}}{1-a_{k+1}}$. □

With Lemma 8, we can now prove Proposition 7.

Proof of Proposition 7. All we need to show is that for all $0 \leqslant k \leqslant d-2$,

$$x_k \geqslant \frac{k+2}{\prod_{i=0}^{k}(1 - a_i)}.$$

We do an induction on k. For $k = 0$, $x_0 = \frac{2}{1-a_0}$ and the claim holds.

By (16), we have that

$$
\begin{aligned}
x_k &= \frac{1+a_k}{1-a_k} x_{k-1} + 1 \\
\text{(by induction hypothesis)} \qquad &\geqslant \frac{(1+a_k)(k+1)}{\prod_{i=0}^{k}(1-a_i)} + 1 = \frac{(1+a_k)(k+1) + \prod_{i=0}^{k}(1-a_i)}{\prod_{i=0}^{k}(1-a_i)} \\
\text{(by Lemma 8)} \qquad &\geqslant \frac{k+2}{\prod_{i=0}^{k}(1-a_i)}.
\end{aligned}
$$

This finishes the induction. □

Notice that the equality in Proposition 7 holds when the trickling down theorem is tight. In this case our bound coincides with Alev and Lau (2020), which is shown in Corollary 5.

4. Decay of relative entropy

In this section we obtain a local-to-global principle for entropy contractions. We obtain bounds for the relative entropy decay of each step of $P_k^{\vee}$, based solely on a property of the local walks on the faces. This answers a question raised by Alev (2020): "... is there a property of the local graphs G_α that would allow us to bound the (modified) log-Sobolev constant of the corresponding chain as opposed to the spectral gap?"

The idea is to follow the argument of Theorem 4 but with every occurrence of $\mathrm{Var}()$ replaced with $\mathrm{Ent}()$. Our initial assumptions for the local walks would now be entropy contraction, and the recursion would be the same. Let us state the theorem for clarity.

Theorem 9. *Let $\mathcal{C}$ be a simplicial complex that satisfies the local inequalities*

$$
\mathrm{Ent}_{\pi_{S,2}}\left(f_S^{(2)}\right) \geqslant s_{k-2}\mathrm{Ent}_{\pi_{S,1}}\left(f_S^{(1)}\right),
$$

for any $2 \leqslant k \leqslant d$, $S \in \mathcal{C}(k-2)$, and $f_S^{(2)} : \mathcal{C}_S(2) \to \mathbb{R}_{\geqslant 0}$, where $\{s_k\}$ are some entropy contraction factors greater than or equal to 1. Then, for any $2 \leqslant k \leqslant d$ and $f^{(k)} : \mathcal{C}(k) \to \mathbb{R}_{\geqslant 0}$, we get the global inequalities

$$
\mathrm{Ent}_{\pi_k}\left(f^{(k)}\right) \geqslant \nu_{k-2}\mathrm{Ent}_{\pi_{k-1}}\left(f^{(k-1)}\right),
$$

where ν_k is recursively defined as in (12).

Proof. The proof is identical to that of Theorem 4 except that $\mathrm{Var}()$ is replaced by $\mathrm{Ent}()$. □

Note that an explicit formula for ν_k is given in (17).

As entropy contraction implies the modified log-Sobolev inequality, we have the following corollary.

Corollary 10. *Let $\mathcal{C}$ be a simplicial complex that satisfies the assumptions of Theorem 9. Then, the following hold:*

- *for any $2 \leqslant k \leqslant r$, $\rho(P_k^{\vee}) \geqslant 1 - \frac{1}{\nu_{k-2}}$;*
- *for any $1 \leqslant k \leqslant r-1$, $\rho(P_k^{\wedge}) \geqslant 1 - \frac{1}{\nu_{k-1}}$.*

Proof. For the down-up walk, combine Theorem 9 with (7). For a proof that applies to both walks, see Cryan et al. (2021). □

We remark that our Theorem 9 is independently obtained by Chen et al. (2020b, Theorem 5.4). For applications of Theorem 9, we refer the reader to Chen et al. (2020b). Although the conditions of Theorem 9 seem restrictive, it is in fact satisfied by certain high-dimensional expanders with bounded marginals, as shown in Chen et al. (2020b).

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