#BIS-Hardness for 2-Spin Systems on Bipartite Bounded Degree Graphs in the Tree Nonuniqueness Region

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Barcelona, Spain

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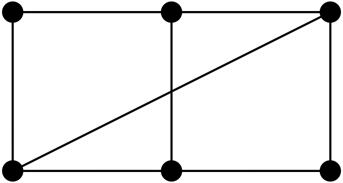
Counting Independent Sets

Independent Sets



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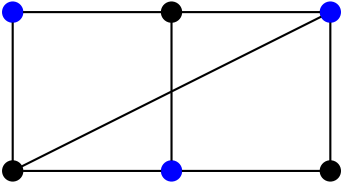

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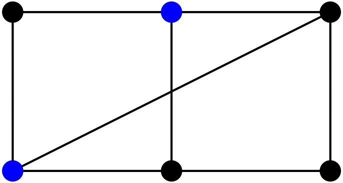

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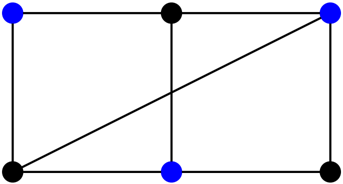

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$$\#\text{IS} = \sum_{\sigma: V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = 1$ if σ induces an independent set and $w(\sigma) = 0$ otherwise.

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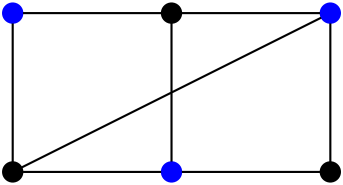


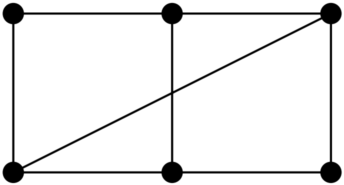
Hardcore gas model:

$$Z_G(\lambda) = \sum_{\sigma: V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = \lambda^{|\sigma|}$ if σ induces an independent set and $w(\sigma) = 0$ otherwise.

The Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$:



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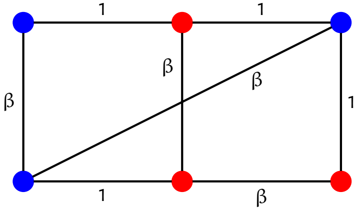

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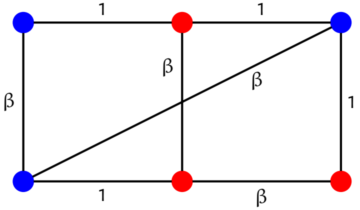

Ising model:

$$Z_G(\beta) = \sum_{\sigma: V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ.

2-Spin Models

Parametrization: edge function $\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$ and vertex weight $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

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Partition function:

$$Z_G(\beta, \gamma, \lambda) = \sum_{\sigma : V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m_0(\sigma)} \gamma^{m_1(\sigma)} \lambda^{n_1(\sigma)}$,

$m_i(\sigma)$ is the number of (i, i) edges under σ,

$n_1(\sigma)$ is the number of 1 vertices under σ.

Gibbs Measure on Infinite Trees

Let $\mathbb{T}_\Delta$ be the infinite Δ-regular tree.

- A Gibbs measure on $\mathbb{T}_\Delta$ is a measure such that for any finite subtree $T \subset \mathbb{T}_\Delta$, the induced distribution on T conditioned on the outer boundary is the same as that given by $\Pr(\sigma) = \frac{w(\sigma)}{Z}$.

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- (Semi-)translation-invariant: invariant under all (parity-preserving) automorphisms of $\mathbb{T}_\Delta$.
- Phase transition: the uniqueness of (semi-)translation invariant Gibbs measures may change as parameters change.
- For anti-ferro ($\beta\gamma < 1$) systems, translation invariant Gibbs measure is always unique, whereas semi-translation invariant ones may not be.

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- On the algorithmic side, there exists an FPTAS for the partition function of the parameter set $(\beta, \gamma, \lambda, \Delta)$ satisfying the uniqueness condition [Weitz 06], [Sinclair, Srivastava, Thurley 12], [Li, Lu, Yin 12, 13].

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- On the hardness side, it is NP-hard to approximate the partition function beyond the uniqueness threshold [Sly 10], [Galanis, Štefankovič, Vigoda 12], [Sly, Sun 12].

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- Algorithms still apply, but hardness results break because reductions are from MAX-CUT.
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  Conjectured to have intermediate complexity in approximation.
- Neither algorithm nor hardness reduction is known for #BIS.

Main Results

Theorem

For all tuples of parameters $(\beta, \gamma, \lambda, \Delta)$ with $\Delta \geqslant 3$ and $\beta\gamma < 1$, if $\mathbb{T}_\Delta$ is in the non-uniqueness region, then approximating $Z_G(\beta, \gamma, \lambda)$ on bipartite graphs with maximum degree Δ is #BIS-equivalent, except for the case $(\beta = \gamma, \lambda = 1)$, which has an FPRAS.

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Corollary

Approximately counting independent sets in bipartite graphs with maximum degree 6 is as hard as without the degree constraint.

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- Moreover, in the $+$ phase, vertices in $T^\pm$ are drawn with probability $p^\pm$. In the $-$ phase, it is reversed.

We call this nearly independent phase correlated spins.

Nearly Independent Phase Correlated Spins - Definition

Given t and ε, let G be drawn from $\mathcal{G}(t, n(t, \varepsilon), \Delta)$. The following should hold with probability at least $3/4$:

1. The phases are roughly balanced, i.e.,

$$\Pr_{G;\beta,\gamma,\lambda}(Y(\sigma) = +) \geqslant \frac{1}{f(t,\varepsilon)} \text{ and } \Pr_{G;\beta,\gamma,\lambda}(Y(\sigma) = -) \geqslant \frac{1}{f(t,\varepsilon)}.$$

2. For a configuration $\sigma : V \to \{0,1\}$ and any $\tau : T \to \{0,1\}$,

$$\left| \frac{\Pr_{G;\beta,\gamma,\lambda}(\sigma|_T = \tau \mid Y(\sigma) = +)}{Q^+(\tau)} - 1 \right| \leqslant \varepsilon \text{ and } \left| \frac{\Pr_{G;\beta,\gamma,\lambda}(\sigma|_T = \tau \mid Y(\sigma) = -)}{Q^-(\tau)} - 1 \right| \leqslant \varepsilon,$$

where Q^+ is the joint distribution where each vertex in $T^\pm$ is drawn independently with probability $p^\pm$, and swapping p^+ and p^- gives Q^-.

Symmetry Breaking

Definition

A tuple of parameters $(\beta, \gamma, \lambda, \Delta)$ supports symmetry-breaking if there is a bipartite graph H whose vertices have degree at most Δ with a distinguished degree-1 vertex v_H such that $\Pr_{H;\beta,\gamma,\lambda}(\sigma_{v_H} = 1) \notin \{0, \lambda/(1+\lambda), 1\}$.

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- On bipartite graphs, anti-ferro Ising models without external fields ($\beta = \gamma < 1$, $\lambda = 1$) can be reduced to ferromagnetic systems, by flipping one side's assignments.
- We showed that all other cases support symmetry breaking.

A Sufficient Condition of #BIS-hardness

General Graphs

If a parameter set $(\beta, \gamma, \lambda, \Delta)$ supports nearly independent phase correlated spins, then

Sly showed a reduction from MAX-CUT to approximating $Z_G(\beta, \gamma, \lambda)$ [Sly 10].

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Bipartite Graphs - Our Result

If a parameter set $(\beta, \gamma, \lambda, \Delta)$ supports both nearly independent phase correlated spins and symmetry breaking, then approximating $Z_G(\beta, \gamma, \lambda)$ is #BIS-equivalent.

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- Non-uniqueness $\Rightarrow$ nearly independent phase correlated spins.
- All parameters except $(\beta = \gamma, \lambda = 1) \Rightarrow$ symmetry breaking.

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- The first step is from #BIS to an Ising model with the edge interaction β and non-uniform external field λ for any $0 < \beta < 1$ and $\lambda \neq 1$.

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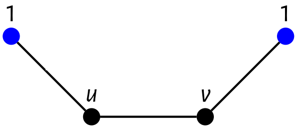

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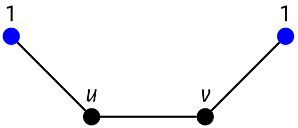

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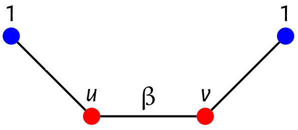

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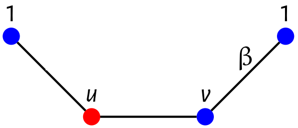

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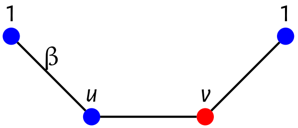

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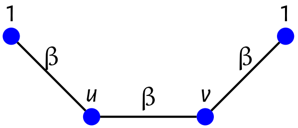

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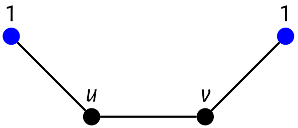

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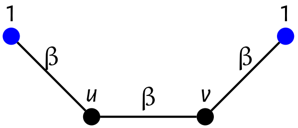
  - $\begin{bmatrix} 1 & 1 \\ 1 & \beta^2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ as β goes to 0.



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    Replace each edge by connecting terminals: $+$ to $+$ and $-$ to $-$.
  - Effective edge interaction is of the Ising type.
- However, like Sly's gadget, we only require the two phases of the gadget to be polynomially balanced. This induces an unpleasant polynomially large external field on each vertex.

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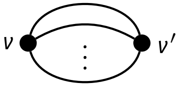
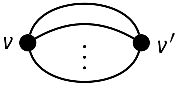
- We want to control the external field on each vertex. Hence we need the two phases to occur with roughly equal probability, i.e. about $1/2$.
  - Construct a new gadget by gluing two gadgets together, and connect many terminals between them. With high probability the two gadgets will have different phases.



Balance the Gadget

- Sly's MAX-CUT reduction works when the two phases occur with probabilities that are bounded below by an inverse polynomial.
- We want to control the external field on each vertex. Hence we need the two phases to occur with roughly equal probability, i.e. about $1/2$.
  - Construct a new gadget by gluing two gadgets together, and connect many terminals between them. With high probability the two gadgets will have different phases.
  - Define the phase of the whole gadget to be the phase of the first. The two phases are balanced as



$$\Pr(+-) = \Pr(-+).$$

Summary

Theorem

For all tuples of parameters $(\beta, \gamma, \lambda, \Delta)$ with $\beta\gamma < 1$ and $\Delta \geqslant 3$, the following holds:

1. If the parameters satisfy strict-uniqueness then there is a FPTAS for the partition function for all graphs [Li, Lu, Yin 13].
2. If the parameters satisfy non-uniqueness then:
   1. it is #SAT-hard to approximate the partition function on graphs [Sly, Sun 12].
   2. it is #BIS-hard to approximate the partition function on bipartite graphs, except when $\beta = \gamma$ and $\lambda = 1$, which admits an FPRAS.

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  - Recent progress on #BIS-hardness has been made based on our results [Liu, Lu, Zhang 14].

Thank You!

Papers and slides available on my homepage:

`www.cs.wisc.edu/~hguo/`