The Complexity of Ising Models with Complex Weights

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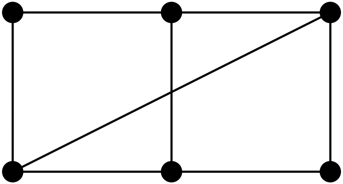
Ann Arbor, MI
Dec 6th 2014

Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$



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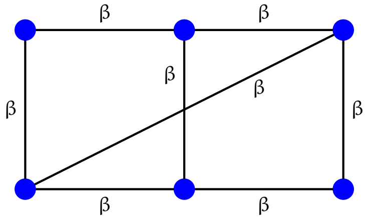

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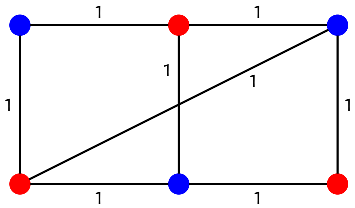

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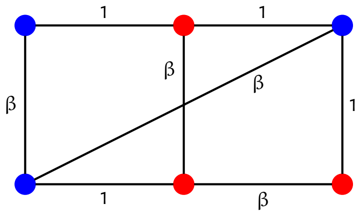

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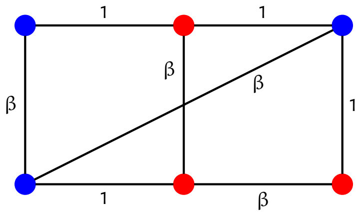

Partition function (normalizing factor):

$$Z_G(\beta) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ.

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In this talk we will focus on approximating $Z_G(\beta)$ for $\beta \in \mathbb{C}$.

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But is the quantum machinery necessary to study $Z_G(\beta)$?

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Hardness of approximating $|z|$ or $\arg(z)$ implies hardness under Ziv's measure.

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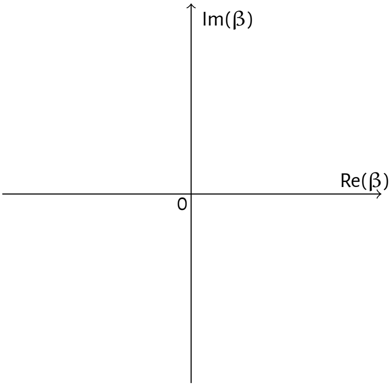

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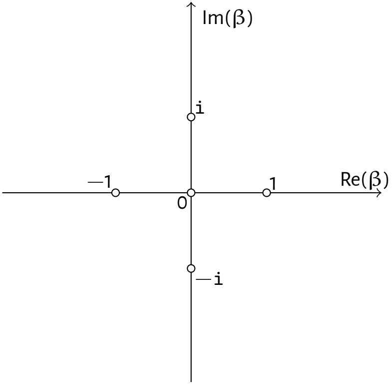

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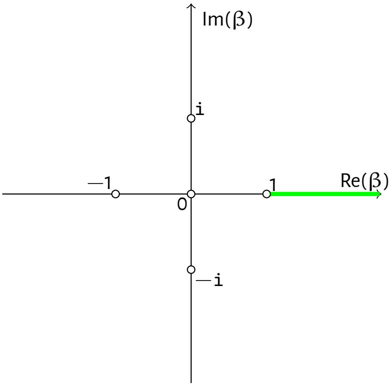

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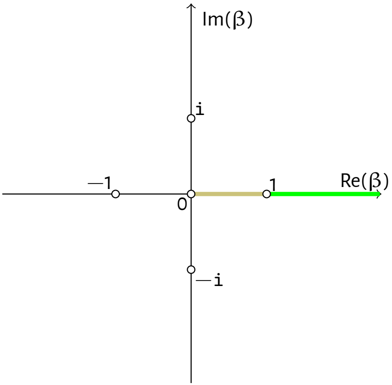

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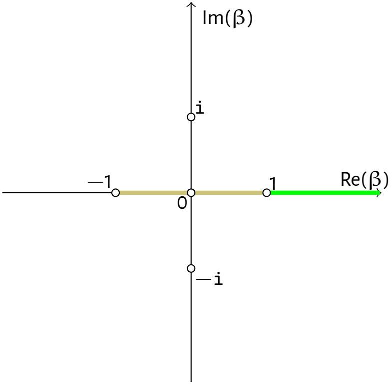

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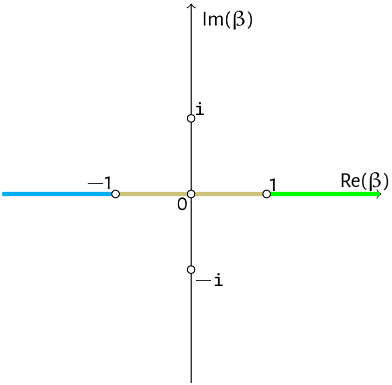

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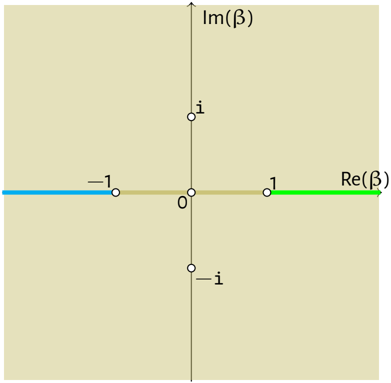

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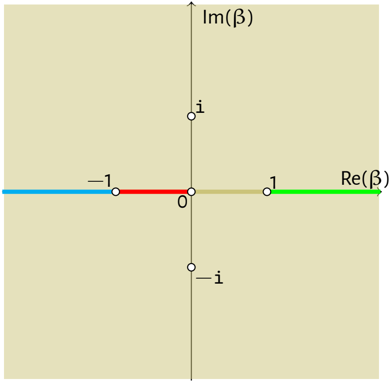

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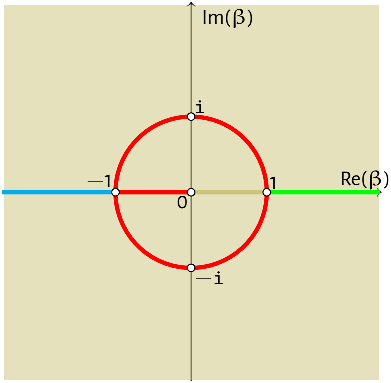

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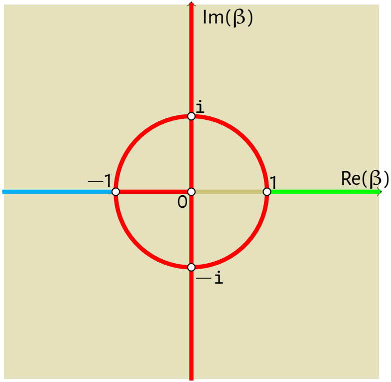

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The key part of the #**P**-hardness proof is a bisection argument.
This idea has been used to show hardness of determining signs of Tutte polynomials (at real points). [Goldberg, Jerrum, 12]

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  Moreover if we can approximate x_0 accurately enough, C can be computed exactly.

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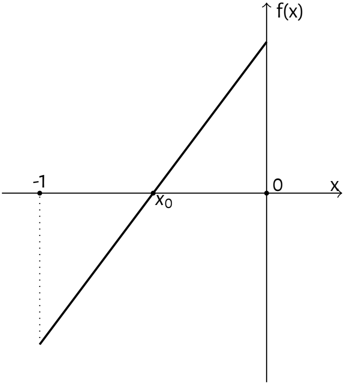

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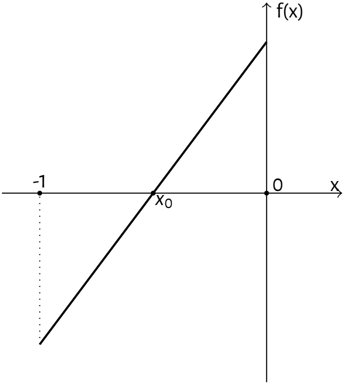

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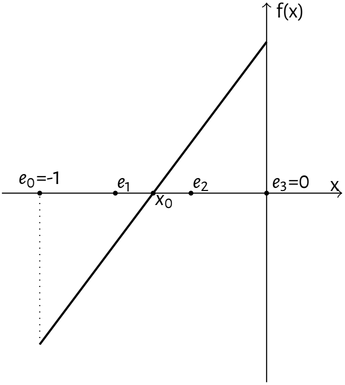

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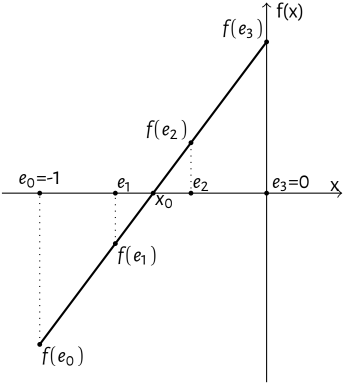

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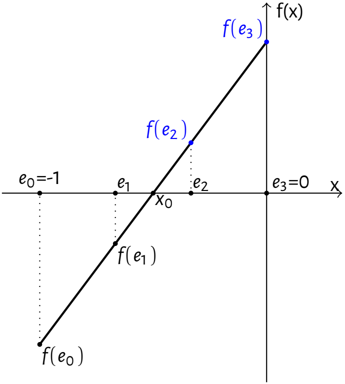

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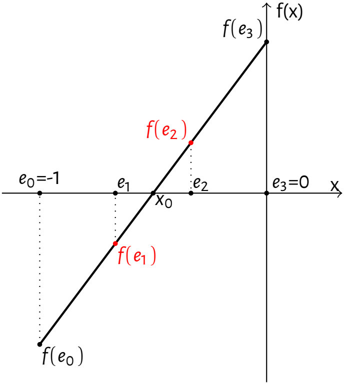
- Otherwise the order may be wrong, but it happens at most once.



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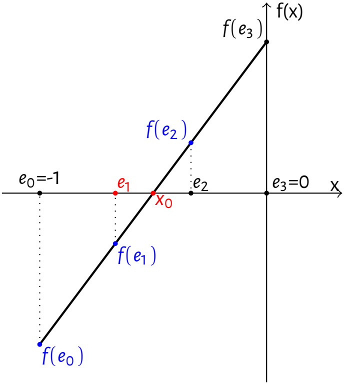

Bisection with an Oracle of Approximating Norms

The oracle returns $|f(x)|$ up to some constant K. Call the approximation $g(x)$.
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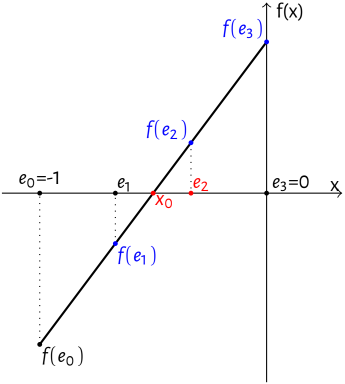

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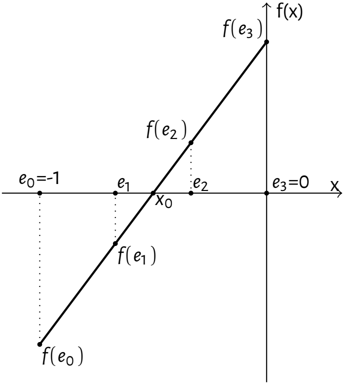

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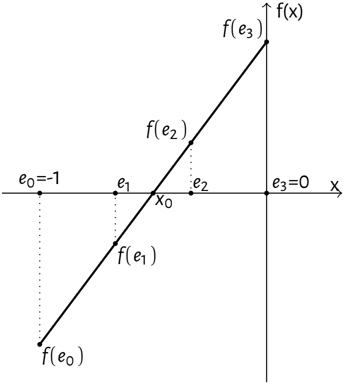

We divide the interval into more subintervals so that we don't need an exact evaluation of $|f(x)|$ at x_0.

Further results

- Hardness results of approximating $\arg(Z_G(\beta))$.

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  We showed that to determine this sign is #**P**-hard over general graphs.
- A complete classification of approximating partition functions of Ising models with external fields, when both the edge weight and the field are roots of unity.

Complex Ising with Fields

Edge weight β, external field λ:

$$Z_G(\beta;\lambda) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}\lambda^{c_1(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ, and $c_1(\sigma)$ is the number of “blue” vertices.

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Theorem

Let β and λ be two roots of unity. Then the following holds:

- If $\beta = \pm 1$, or $\beta = \pm \mathtt{i}$ and $\lambda \in \{1, -1, \mathtt{i}, -\mathtt{i}\}$, $Z_G(\beta;\lambda)$ can be computed exactly in polynomial time.
- Otherwise $|Z_G(\beta;\lambda)|$ is #**P**-hard to approximate.

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- So we can assume $|Z_G(\beta)| \neq 0$. All we need is a lower bound of $|Z_G(\beta)|$.
  (Even an exponential one suffices.)
- If β is rational, this is straightforward by a granularity argument.
  If β is algebraic, we need to use some basic transcendental number theory.

Thank You!

Papers and slides available on my homepage:
`www.cs.wisc.edu/~hguo/`