A Complete Dichotomy Rises from the Capture of Vanishing Signatures

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Contents

Counting problems

Computational Counting problems appear often in statistical physics, machine learning, quantum computation, information theory, and so on. The quantity to be computed is usually expressed as a sum of products.

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- The expectation of any random variable;
- Approximate an integral by a weighted sum;
- Classical simulation of quantum circuits;
- Partition functions.

  Ising model, Potts model, Hard-core gas model, …

Partition functions

Let us take a closer look at the partition functions.

- Ising model (without an external field):

$$\sum_{\sigma:V\mapsto\{+,-\}} \beta^{n(\sigma)},$$

where $n(\sigma)$ is the number of $(+,+)$ and $(-,-)$ neighbours in the graph given σ.

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- We can rewrite it in the following form:

$$\sum_{\sigma : V \mapsto \{0,1\}} \prod_{(i,j) \in E} f_{\text{ISING}}(\sigma(i), \sigma(j)),$$

  where $f_{\text{ISING}}(0,0) = f_{\text{ISING}}(1,1) = \beta, \quad f_{\text{ISING}}(0,1) = f_{\text{ISING}}(1,0) = 1.$

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where $f_{\text{IS}}(0,0) = f_{\text{IS}}(1,0) = f_{\text{IS}}(0,1) = 1, \quad f_{\text{IS}}(1,1) = 0,$

$g_{\text{IS}}(0) = 1$ and $g_{\text{IS}}(1) = \lambda$.

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$$\sum_{\sigma : E \mapsto \{0,1\}} \prod_{v \in V} f_{\text{PM}}(\sigma \mid_{E(v)}),$$

where $\sigma \mid_{E(v)}$ is the assignment σ restricted to the set $E(v)$ of incident edges of v, and f_{PM} is the Exact-One function.

Common features

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- Functions take assignments on adjacent edges/vertices as inputs.
- Quantity to compute is an exponential sum over all possible assignments.

Frameworks

Counting problems are often parameterized by constraint functions. Frameworks specify where to put the functions and to sum over what assignments.

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (#CSP)
3. Holant Problems

The expressive power is increasing in order.

Instance - signature grid

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where each vertex is labeled by a function $f_v \in \mathcal{F}$, and $\pi : V \to \mathcal{F}$ is the labelling.

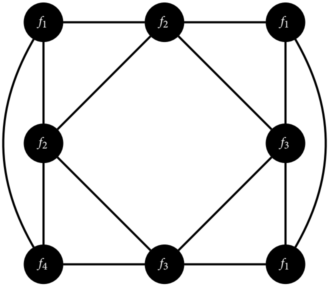

Figure: A signature grid

Holant problems

The Holant problem on instance Ω is to evaluate

$$\text{Holant}_\Omega = \sum_{\sigma:E\mapsto\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E(v)}),$$

a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

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When the function is EXACT-ONE, the Holant counts perfect matchings in G.

- Such a function is symmetric. The output only depends on the Hamming weight of the input.
- List a symmetric function f by the Hamming weights: $[f_0, f_1, \ldots, f_n]$.

Some examples

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- EXACT-ONE: $[0, 1, 0, \ldots, 0]$. The Holant counts perfect matchings.
- AT-MOST-ONE: $[1, 1, 0, \ldots, 0]$. The Holant counts matchings.
- What about $f = [3, 0, 1, 0, 3]$?

Holant([3, 0, 1, 0, 3])

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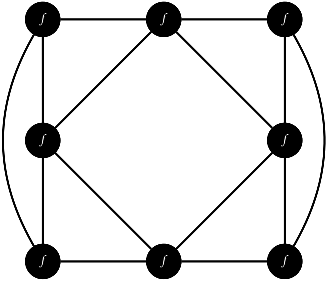

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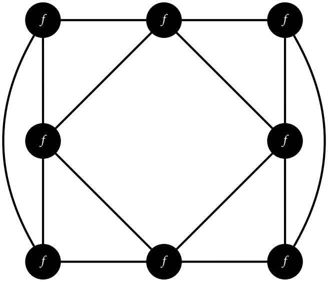

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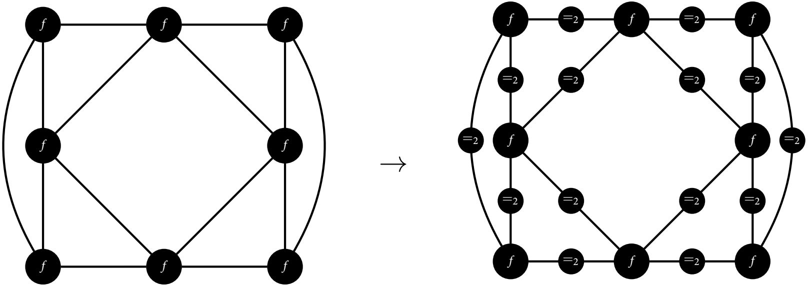

Holographic transformation

For a 2-by-2 nonsingular matrix T, two functions f and g of arities m and n respectively, Valiant's Holant theorem states

$$\text{Holant}\,(f \mid g) = \text{Holant}\left(fT^{\otimes m} \mid (T^{-1})^{\otimes n} g\right)$$

This is what we call a holographic transformation. Here f is treated as a row vector of length 2^m and g as a column vector of length 2^n.

Holant([3, 0, 1, 0, 3])

We apply the transformation $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ to $\text{Holant}\left(=_2 \mid [3, 0, 1, 0, 3]\right)$.

- $(=_2)Z^{\otimes 2} = [0, 1, 0]$, which we denote by $\neq_2$.
- $(Z^{-1})^{\otimes 4}([3, 0, 1, 0, 3]) = 2[0, 0, 1, 0, 0]$. The constant does not affect the complexity.

Therefore,

$$\text{Holant}\left(=_2 \mid [3, 0, 1, 0, 3]\right) = \text{Holant}\left(\neq_2 \mid 2[0, 0, 1, 0, 0]\right).$$

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Two integer weighted problems are equivalent via a complex holographic transformation.

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- Real-weighted dichotomy [Huang and Lu '12].
- Tractable cases:
  - Equivalent to a problem on graphs of bounded degree 2;
  - Equivalent to a tractable #CSP problem (via a holographic transformation).

Our Contribution

Theorem

Let $\mathcal{F}$ be a set of complex-weighted Boolean symmetric functions. Then $\operatorname{Holant}(\mathcal{F})$ *is either tractable or #P-hard.*

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Let $\mathcal{F}$ be a set of complex-weighted Boolean symmetric functions. Then $\text{Holant}(\mathcal{F})$ *is either tractable or #P-hard.*

- A new class of tractable functions: vanishing signatures.
- A clear characterization regarding cases that can be transformed into tractable #CSPs.
- Everything else is hard.

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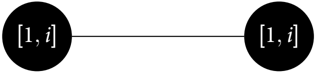

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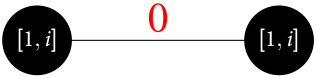

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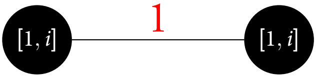

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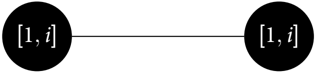
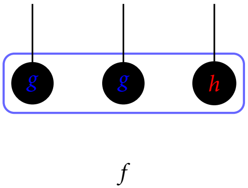


$$1 \cdot 1 + i \cdot i = 0$$

Vanishing

- We can view several unary signatures as a new one, which we call degenerate. It is the tensor product of the unary signatures. For example,

$$f = g \otimes g \otimes h.$$



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- Any degenerate signature containing more than half $[1, i]$'s is vanishing. For example,

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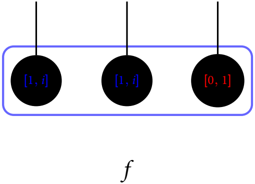

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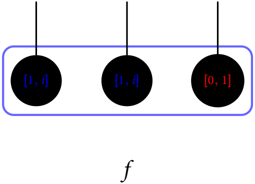

- However, such signatures are not symmetric. We need to introduce an operation of symmetrization.

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Symmetrization

Let S_n be the symmetric group of degree n. Then for positive integers t and n with $t \leq n$ and unary signatures $v, v_1, \ldots, v_{n-t}$, we define

$$\mathrm{Sym}_n^t(v; v_1, \ldots, v_{n-t}) = \sum_{\pi \in S_n} \bigotimes_{k=1}^{n} u_{\pi(k)},$$

where the ordered sequence

$$(u_1, u_2, \ldots, u_n) = (\underbrace{v, \ldots, v}_{t \text{ copies}}, v_1, \ldots, v_{n-t}).$$

Examples

For example,

$$\begin{aligned}\mathrm{Sym}_3^2([1,i];[0,1]) &= 2[0,1]\otimes[1,i]\otimes[1,i] + 2[1,i]\otimes[0,1]\otimes[1,i] + 2[1,i]\otimes[1,i]\otimes[0,1]\\ &= 2[0,1,2i,-3].\end{aligned}$$

Vanishing degrees

Definition

A nonzero symmetric signature f of arity n has positive vanishing degree $k \geq 1$, which is denoted by $\mathrm{vd}^+(f) = k$, if $k \leq n$ is the largest positive integer such that there exists $n - k$ unary signatures $v_1, \ldots, v_{n-k}$ satisfying

$$f = \mathrm{Sym}_n^k([1, i]; v_1, \ldots, v_{n-k}).$$

If f cannot be expressed as such a symmetrization form, we define $\mathrm{vd}^+(f) = 0$.

If f is the all zero signature, define $\mathrm{vd}^+(f) = n + 1$.

We define $\mathrm{vd}^-(f)$ similarly, using $-i$ instead of i.

Characterization

Theorem

A signature f is vanishing if and only if

$$2\,\mathrm{vd}^{\sigma}(f) > \mathrm{arity}(f)$$

for $\sigma = +$ *or* $-$.

This result also generalizes to a set of signatures.

- $\mathrm{vd}^{+}([0, 1, 2i, -3]) = 2$, so $[0, 1, 2i, -3]$ is vanishing.
- $\mathrm{vd}^{+}([1, 0, 1]) = \mathrm{vd}^{-}([1, 0, 1]) = 1$. It is not vanishing.

Related tractable cases

- Vanishing signatures are by themselves tractable.
- Some unary and binary (non-vanishing) signatures can be combined with them and remain tractable.
- Technically there are two categories, but the basic idea is that for these problems a given instance is either vanishing or of bounded degree 2.

Thank you!