Approximation via Correlation Decay when Strong Spatial Mixing Fails

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Rome, Italy

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Independent sets

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An independent set is a subset of V such that no two are adjacent.

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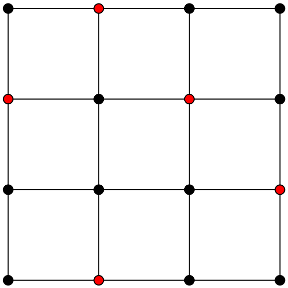

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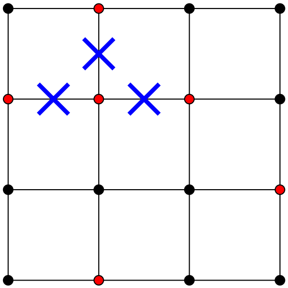

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Alternatively, consider the following distribution:

$$\pi(I) \propto \lambda^{|I|}$$

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Partition function $Z = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}$.

In particular, if $\lambda = 1$, $Z = |\mathcal{I}(G)|$.

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Approximate counting weighted independent sets

(or approximate Z for the Hardcore model)

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Counting independent sets

Specialize to approximate counting independent sets (fix $\lambda = 1$):

For G with a degree bound Δ:

FPTAS	**NP**-hard
$\Delta \leqslant 5$	$\Delta \geqslant 6$

($\Delta = 5$ is the largest integer so that $\lambda_c(\Delta) \geqslant 1$)

- Algorithm: [Weitz 06]
- Hardness: [Sly 10]

Independent sets in hypergraphs

Hypergraph $H = (V, F)$, where a hyperedge $e \in F$ is a subset of V.

Independent set I: $I \subseteq V$ and $\forall e \in F$, $e \not\subset I$. (NOT-ALL-IN)

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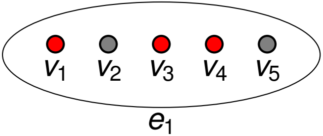

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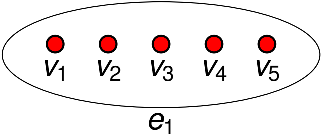

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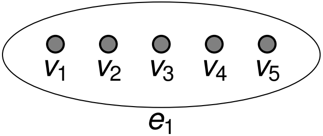

$2^5 - 1$ many independent sets

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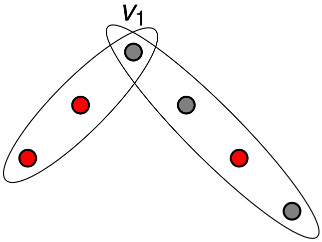

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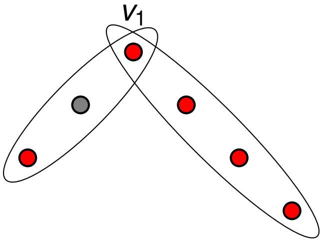

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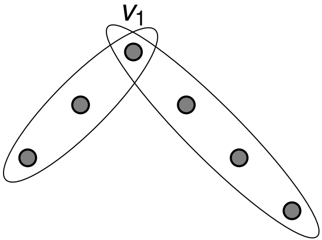

$2^2 \cdot 2^3 + (2^2 - 1)(2^3 - 1)$

many independent sets

Monotone CNF

Independent Sets in Hypergraphs $\Leftrightarrow$

Satisfying assignments of monotone CNF formulas.

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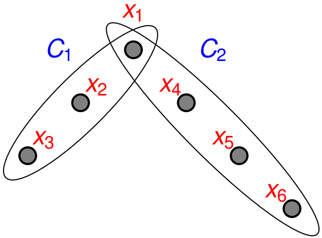

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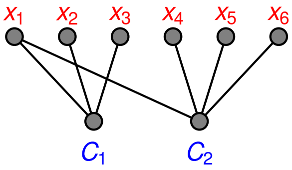

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Bounded occurrences

Name #HYPERINDSET(Δ, k).

Instance A hypergraph H with maximum degree at most Δ where each hyperedge has cardinality (arity) at least k.

Output The number Z_H of independent sets in H.

Previously

Based on Markov chain Monte Carlo:

- There is a FPRAS for #HYPERINDSET(Δ, k) if $k \geqslant \Delta + 2$.

  [Dyer Greenhill 00], [Borderwich, Dyer, Karpinski 06, 08].

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- There is a FPTAS for $\#\text{HyperIndSet}(\Delta, k)$ if $\Delta \leqslant 5$ for any integer $k \geqslant 2$.
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$\#\text{HYPERINDSET}(\Delta, 2)$ is at least as hard as counting independent sets.

Hence FPTAS for $k = 2$, $\Delta \leqslant 5$ is optimal. ($\Delta \geqslant 6$ is **NP**-hard [Sly 10].)

Our results

Theorem

There is a FPTAS for $\#\text{HYPERINDSET}(\Delta, k)$ *if*

1. $\Delta = 6$ *and* $k \geqslant 3$;
2. *For* $\Delta \geqslant 200$, $k \geqslant 1.66\Delta$.

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- For $\Delta = 6$, $k = 3$ is optimal as #HYPERINDSET$(6, 2)$ is **NP**-hard [Sly 10].
- $k \geqslant \Delta$ only slightly improves $k \geqslant \Delta + 2$ [Borderwich, Dyer, Karpinski 08], but this improvement is essential for our application of counting dominating sets in regular graphs.

Hardness

Theorem

For any integer $\Delta \geqslant 5 \cdot 2^{k/2}$, it is **NP***-hard to approximate* #HYPERINDSET(Δ, k)*, even within an exponential factor.*

FPTAS: $\Delta \leqslant k$

NP-hard: $\Delta \geqslant 5 \cdot 2^{k/2}$

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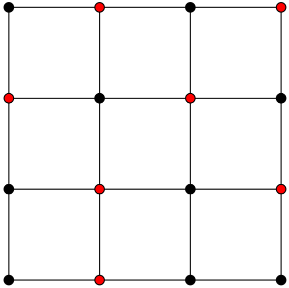

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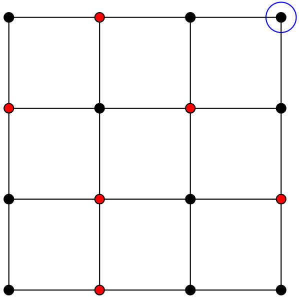

Not dominating

Counting dominating sets

Name #REGDOMSET(Δ).

Instance A Δ-regular graph G.

Output The number of dominating sets in G.

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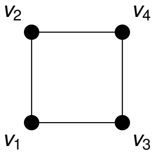

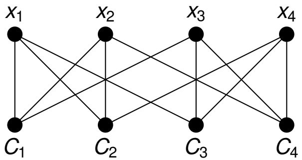

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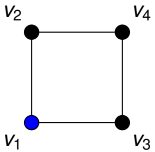

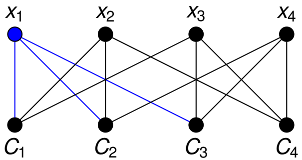

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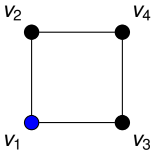

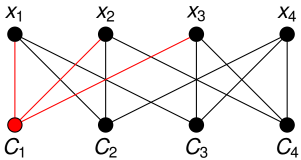

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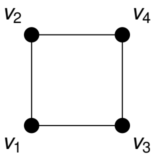

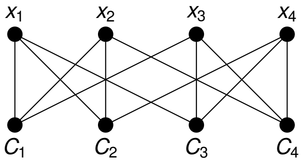

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Approximately counting dominating sets is **NP***-hard in graphs with degree bound* $\Delta \geqslant 18$*, even within an exponential factor.*

The Algorithm

A recursion for counting independent sets

$$P_G(v) = \frac{|\{I \in \mathcal{I} \mid v \notin I\}|}{|\mathcal{I}|} = \frac{Z(G-v)}{Z(G)}$$

$$= \frac{Z(G-v)}{Z(G-v) + Z(G-v-N(v))}$$

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Suppose $N(v) = \{v_1, \ldots, v_d\}$.

$$\frac{Z(G-v-N(v))}{Z(G-v)} = \frac{Z(G-v-v_1)}{Z(G-v)} \cdot \frac{Z(G-v-v_1-v_2)}{Z(G-v-v_1)} \cdot \cdots \cdot \frac{Z(G-v-N(v))}{Z(G-v-(N(v)-v_d))}$$

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$$= P_{G_1}(v_1) \cdot P_{G_2}(v_2) \cdots P_{G_d}(v_d)$$

Here $G_i = G - v - v_1 - \cdots - v_{i-1}$.

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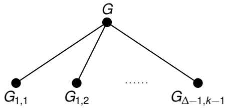
- The recursion forms a computation tree.
- We stop the recursion after $O(\log n)$ many steps.
- The main task is to bound the error.



Strong spatial mixing

SSM: Let σ_Λ and τ_Λ be two partial configurations on $\Lambda \subseteq V$.

Let S be the set where σ_Λ and τ_Λ differ.

$$|p_v^{\sigma_\Lambda} - p_v^{\tau_\Lambda}| \leqslant \exp(-\Omega(\mathrm{dist}(v, S)))$$

Roughly speaking, the influence of the boundary decays exponentially,

even with some vertices fixed within the radius.

SSM for hypergraph independent sets

In the computation tree, hyperedge sizes will decrease.

Eventually, the size may go down to 2.

Hence SSM does not hold for independent sets in hypergraphs when $\Delta \geqslant 6$.

(SSM does not hold for independent sets in graphs if $\Delta \geqslant 6$.)

This is why [Liu, Lu 15] can only do $\Delta \leqslant 5$.

Beyond strong spatial mixing

Our contribution is to provide a way to analyze correlation decay beyond the strong spatial mixing bound.

- Larger hyperedges have better decay.
- Keep track of the total “deficits” of sub-instances.
- Amortized analysis — SSM is worst case.

Main difficulty — bound the decay rate

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a optimisation problem with $(\Delta - 1)(k - 1)$ many variables.

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Decay Rate

$$\kappa^{d,\mathbf{k}}(\mathbf{r}) := \frac{1}{\psi - F(\mathbf{r})^{\chi}} \sum_{i=1}^{d} \alpha^{-l_{k_i-1}} \frac{\prod_{j=1}^{k_i-1} \frac{r_{i,j}}{1+r_{i,j}}}{1 - \prod_{j=1}^{k_i-1} \frac{r_{i,j}}{1+r_{i,j}}} \sum_{j=1}^{k_i-1} \delta^{c_{i,j}} \frac{\psi - r_{i,j}^{\chi}}{1 + r_{i,j}},$$

where

$$F(\mathbf{r}) = \prod_{i=1}^{d} \left(1 - \prod_{j=1}^{k_i-1} \frac{r_{i,j}}{1 + r_{i,j}}\right)$$

$$c_{i,j} = b_2(k-2) + s_{\min(i,d-b_2)} - \max(0, b'_k - i) - (j-1)(\Delta - 1)\mathbf{1}_{i \leqslant d-b_2}.$$

Open questions

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Thank You!

Full version: `arxiv.org/abs/1510.09193`