Counting Hypergraph Colourings in the Local Lemma Regime

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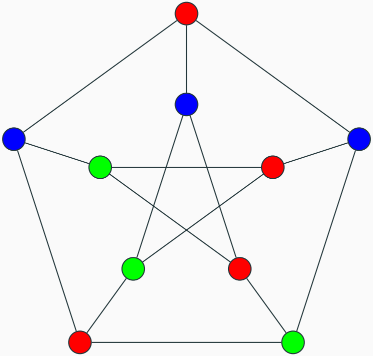
Chinese Academy of Science, Dec 26 2017

Colourings

Graph (proper) colouring

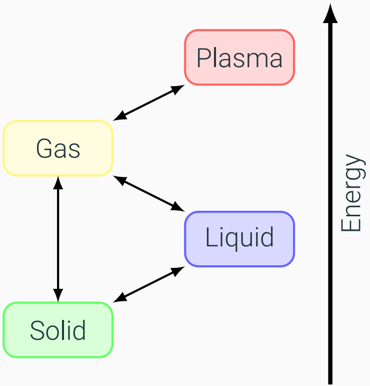


3-colouring of the Petersen graph

Phase transitions

Phase transitions:
as some parameter changes, macroscopic behaviours of the whole system change drastically.

E.g. ice $\to$ water $\to$ water vapor



Computational phase transitions

The complexity of determining whether a graph is q-colourable:

- $q = 1, 2$: easy;
- $q \geqslant 3$: **NP**-hard.

What about graphs with maximum degree Δ?

- $q \geqslant \Delta + 1$: always colourable;
- $q \geqslant \Delta - \sqrt{\Delta} + 3$: polynomial-time (Molloy, Reed '01 '14);
- $q \leqslant \Delta - \sqrt{\Delta} + c$ $(c = 1.5 + O(\Delta^{-0.5}))$: **NP**-hard (Embden-Weinert, Hougardy, and Kreuter '98).

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Properly colour a planar graph

Things can be more complicated!

On a planar graph, determining whether a graph is q-colourable:

- $q = 2$: easy;
- $q = 3$: **NP**-hard (Dailey '80);
- $q = 4$: quadratic time (Four colour theorem) by Robertson, Sanders, Seymour, and Thomas (1996);
- $q \geqslant 5$: linear time (much simpler proof) (RSST '96).

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Randomly colour a graph

What about generate a random proper colouring?

- $q > 2\Delta$: rapid mixing of Glauber dynamics by Jerrum (1995);
- $q > \frac{11}{6}\Delta$: rapid mixing of WSK dynamics by Vigoda (2000);
- $q < \Delta$ (even q) : **NP**-hard by Galanis, Štefankovič, and Vigoda (2015);

It is conjectured that there is a threshold and $q_c = \Delta + 1$. This is the uniqueness threshold of Gibbs measures in an infinite Δ-regular tree (namely a Bethe lattice), by Jonasson (2002).

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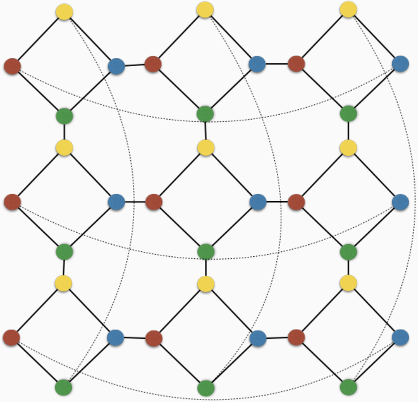
FROZEN

Sometimes you just cannot let it go.

$$q = \Delta + 1 = 4$$

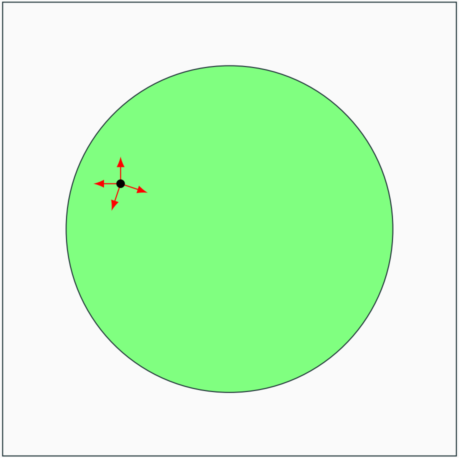


credit: Chihao Zhang

Disconnected state space

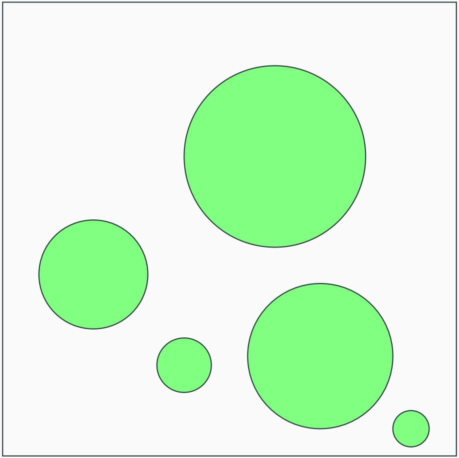
Markov chain is a random walk in the solution space.

(The solution space has to be connected!)



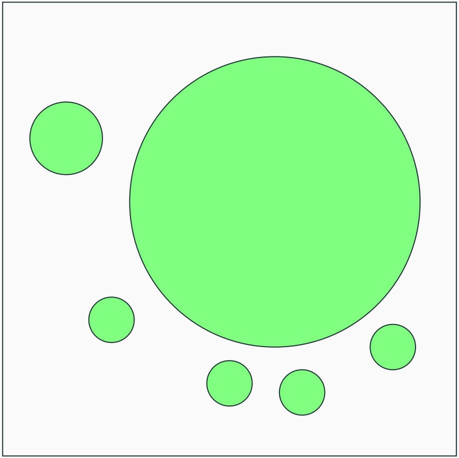
Disconnected state space

A disconnected state space is really bad.



Disconnected state space

There is still some hope if a giant component dominates.



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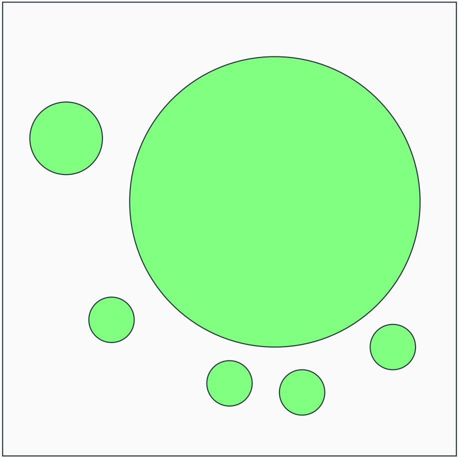
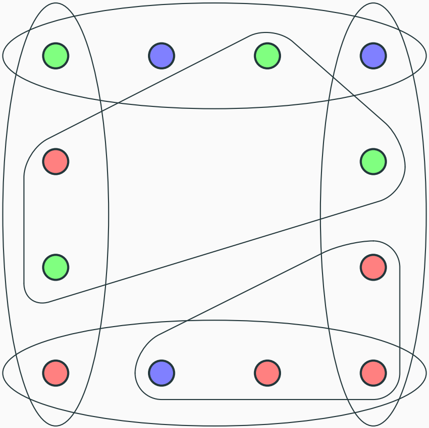
Approximately counting 4-colourings in cubic graphs
by Lu, Yang, Zhang, Zhu (2017). (Not via Markov chains!)



What about hypergraphs?



A proper hypergraph colouring is one where no edge is monochromatic.

Previous results

For k-uniform hypergraphs, Bordewich, Dyer, and Karpinski (2006) show that Glauber dynamics is rapidly mixing if

$$k \geqslant 4 \text{ and } q > \Delta \qquad \text{or} \qquad k = 3 \text{ and } q > 1.5\Delta.$$

However, Lovász local lemma implies that there exists a proper colouring if $q > 2e\Delta^{1/(k-1)}$.

Frieze and Melsted (2011) showed that if $q \ll \Delta$, then there exists a colouring so that no move is possible ("frozen").

Frieze and Anastos (2017) showed that Glauber dynamics still converges rapidly if the hypergraph is simple and $q > \max\{C_k \log n, 500k^3\Delta^{1/(k-1)}\}$.

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Our results

Theorem

For integers $\Delta \geqslant 2$, $k \geqslant 28$, and $q > 315\Delta^{\frac{14}{k-14}}$, there is an FPTAS for q-colourings in k-uniform hypergraphs with maximum degree Δ.

Theorem

For integers $\Delta \geqslant 2$, $k \geqslant 28$, and $q > 798\Delta^{\frac{16}{k-16/3}}$, there is also an almost-uniform polynomial-time sampler.

Our approach is based on a result by Moitra (2017), whose original approach in this setting would require $k > C \log \Delta$ in addition.

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Tensor perspective

Each vertex is a EQUALITY tensor, q-dimensional, order $\leqslant \Delta$.

Each hyperedge is a COLOURING tensor, q-dimensional, order k.

Thus, the hypergraph can be viewed as a bipartite tensor network. The total number of proper colourings is the contraction of the whole tensor network.

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Moitra's approach

(with our modifications)

Lovász local lemma

Let $H = (V, \mathcal{E})$ be the hypergraph, and $\Gamma(e)$ be neighbourhood of $e \in \mathcal{E}$.

Theorem (Lovász '77)

If there exists an assignment $x : \mathcal{E} \to (0, 1)$ such that for every $e \in \mathcal{E}$ we have

$$\Pr(e \textit{ is monochromatic}) \leqslant x(e) \prod_{e' \in \Gamma(e)} (1 - x(e')), \tag{1}$$

then a proper colouring exists.

Normally we set $x(e) = \frac{1}{k\Delta}$ and the condition becomes $q > (ek\Delta)^{\frac{1}{k-1}}$.

Let $\mu_{\mathcal{C}}(\cdot)$ be the Gibbs (uniform) distribution on all proper colourings.

Theorem (Haeupler, Saha, and Srinivasan '11)

If (1) holds for every $e \in \mathcal{E}$, then for any event B, it holds that

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Locally uniformity

Lemma

If $t \geqslant k$ *and* $q \geqslant (et\Delta)^{\frac{1}{k-1}}$*, then for any* $v \in V$ *and any colour* $c \in [q]$*,*

$$\frac{1}{q}\left(1 - \frac{1}{t}\right) \leqslant \Pr_{\sigma\sim\mu_{\mathcal{C}}}(\sigma(v) = c) \leqslant \frac{1}{q}\left(1 + \frac{4}{t}\right).$$

We use this lemma with $t \approx \Delta^C$ at various places with various C. Recall that our assumption is of the form $q \geqslant C'\Delta^{\frac{C''}{k}}$.

Under $\mu_{\mathcal{C}}$, all vertices are very close to uniform.

The use of this lemma is when some vertices are already coloured, namely $\mu_{\mathcal{C}}$ conditioned on a partial colouring. Replace k with the minimum number of uncoloured vertices among all remaining hyperedges.

A good start, but not enough. The goal is $\frac{\varepsilon}{n}$-approximation of the marginals.

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Coupling

Let $\mathcal{C}_i$ be the set of colourings where v is coloured c_i, and μ_i be uniform over $\mathcal{C}_i$. We want to couple $\mathcal{C}_1$ and $\mathcal{C}_2$.

Start: $V_1 = \{v\}$, $V_{col} = \{v\}$. Maintain $V_2 = V \setminus V_1$.

Body:

1. For any hyperedge e intersecting both V_1 and V_2, let u be its first vertex. Couple u maximally assuming we know its marginal probabilities.
2. Remove all hyperedges that are satisfied in both copies.
3. If an edge has k_1 vertices coloured, put other vertices in V_1 (failed) and remove it.

Stop: all hyperedges intersecting both V_1 and $V_2 \setminus V_{col}$ are satisfied.

If $q > C\Delta^{\frac{3}{k-k_1}}$, then the coupling stops in $O(\log n)$ steps with probability $1 - O\left(\frac{1}{n^c}\right)$.

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Coupling tree

Coupling tree $\mathcal{T}$: each node is a pair of partial colourings (x, y).

The children of (x, y) are all q^2 ways to extend them to the next vertex.

All information can be recovered by simulating the coupling from the start.

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A SAMPLER

Let $q_i = \frac{|\mathcal{C}_i|}{|\mathcal{C}|}$ for $i = 1, 2$.

A sampler $\mathbb{S}$:

1. Sample (X, Y) using the coupling;
2. Uniformly output a colouring in $\mathcal{C}_X$ with probability $\frac{q_1}{q_1+q_2}$. Otherwise uniformly output a colouring in $\mathcal{C}_Y$.

The output of $\mathbb{S}$ is uniform over $\mathcal{C}_1 \cup \mathcal{C}_2$.

Linear program

We cannot really run the coupling. Instead, we set up a linear program. The variables are to mimic:

$$p^x_{x,y} = q_1 \cdot \frac{|\mathcal{C}_1 \cup \mathcal{C}_2|}{|\mathcal{C}_x|} \cdot \mu_{cp}(x,y);$$

$$p^y_{x,y} = q_2 \cdot \frac{|\mathcal{C}_1 \cup \mathcal{C}_2|}{|\mathcal{C}_y|} \cdot \mu_{cp}(x,y);$$

$$0 \leqslant p^x_{x,y}, p^y_{x,y} \leqslant 1.$$

The meaning of $p^x_{x,y}$ is, for any $\sigma \in \mathcal{C}_x$,

"conditioned on $\mathbb{S}$ outputting σ, the coupling reaches (x,y)".

This definition is independent from σ.

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$$p^{y}_{x,y} = q_2 \cdot \frac{|\mathcal{C}_1 \cup \mathcal{C}_2|}{|\mathcal{C}_y|} \cdot \mu_{cp}(x,y);$$

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The meaning of $p^{x}_{x,y}$ is, for any $\sigma \in \mathcal{C}_x$,

“conditioned on $\mathbb{S}$ outputing σ, the coupling reaches (x,y)”.

This definition is independent from σ.

Constraints 1

From the definition: $\frac{p^x_{x,y}}{p^y_{x,y}} = \frac{q_1}{q_2} \cdot \frac{|\mathcal{C}_y|}{|\mathcal{C}_x|}$.

If (x, y) is a leaf in $\mathcal{T}$, then we can compute $\frac{|\mathcal{C}_x|}{|\mathcal{C}_y|}$ in time $\exp(|V_1|)$.

Constraints 1: For every leaf (x, y), we have the constraints:

$$\underline{r} \leqslant \frac{p^x_{x,y}}{p^y_{x,y}} \cdot \frac{|\mathcal{C}_x|}{|\mathcal{C}_y|} \leqslant \overline{r}.$$

Here $\underline{r}$ and $\overline{r}$ are our guessed lower and upper bounds for $\frac{q_1}{q_2}$.

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Constraints 2

Constraints 2: For the root $(x_0, y_0) \in \mathcal{T}$, we have

$$p^{x_0}_{x_0,y_0} = p^{y_0}_{x_0,y_0} = 1.$$

Moreover, for every non-leaf $(x, y) \in \mathcal{T}$, let u be the next vertex to couple. We have:

$$\text{for every } c \in [q], \quad p^{x}_{x,y} = \sum_{c' \in [q]} p^{x^{u \leftarrow c}}_{x^{u \leftarrow c}, y^{u \leftarrow c'}};$$

$$\text{for every } c \in [q], \quad p^{y}_{x,y} = \sum_{c' \in [q]} p^{y^{u \leftarrow c}}_{x^{u \leftarrow c'}, y^{u \leftarrow c}}.$$

Recover the marginals

Due to **Constraints 2**, a simple induction shows that for every $\sigma \in \mathcal{C}_1$,

$$\sum_{(x,y)\in\mathcal{L}(T):\ \sigma\models x} p^x_{x,y} = 1\,.$$

Rewrite $|\mathcal{C}_1|$:

$$\begin{aligned}|\mathcal{C}_1| = \sum_{\sigma\in\mathcal{C}_1} 1 &= \sum_{\sigma\in\mathcal{C}_1} \sum_{(x,y)\in\mathcal{L}(\mathcal{T}):\sigma\models x} p^x_{x,y} \\ &= \sum_{(x,y)\in\mathcal{L}(\mathcal{T})} \sum_{\sigma\models x} p^x_{x,y} \\ &= \sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^x_{x,y}\,|C_x|\,.\end{aligned}$$

Similar equalities hold on the Y side, implying:

$$\frac{q_1}{q_2} = \frac{|\mathcal{C}_1|}{|\mathcal{C}_2|} = \frac{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^x_{x,y}\,|C_x|}{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^y_{x,y}\,|C_y|}\,.$$

Recover the marginals (cont.)

$$\frac{q_1}{q_2} = \frac{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^x_{x,y}\,|C_x|}{\sum_{(x,y)\in\mathcal{L}(\mathcal{T})} p^y_{x,y}\,|C_y|}$$

Recall **Constraints 1**. For any $(x, y) \in \mathcal{L}(\mathcal{T})$,

$$\underline{r} \leqslant \frac{p^x_{x,y}\,|C_x|}{p^y_{x,y}\,|C_y|} \leqslant \overline{r}.$$

It implies that

$$\underline{r} \leqslant \frac{q_1}{q_2} \leqslant \overline{r}.$$

Constraints 3

Unfortunately, the whole linear program is exponentially large, but the coupling stops at $O(\log n)$ size whp. If we truncate at $O(\log n)$ levels, the error should be small, due to local uniformity.

Constraints 3: For every $c, c' \in [q]$ that $c \neq c'$:

$$p^{x^{u \leftarrow c}}_{x^{u \leftarrow c}, y^{u \leftarrow c'}} \leqslant \frac{5}{t} \cdot p^{x}_{x,y};$$
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Truncation error

Recall that

$$|\mathcal{C}_1| = \sum_{\sigma \in \mathcal{C}_1} \sum_{(x,y) \in \mathcal{L}(\mathcal{T}) : \sigma \models x} p_{x,y}^{x}.$$

The truncation error from a particular $\sigma \in \mathcal{C}_1$ comes from conditioned on outputing σ, the coupling lasts too long.

Such "bad" colouring does exist (all early vertices are monochromatic).

We prove two things:

1. The fraction of "bad" colourings is small;
2. For every "good" colouring, the truncation error is small because of **Constraints 3**.

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Counting and sampling

So far we are calculating the marginal probability, which requires that there are sufficiently many vertices in all hyperedges.

- For approximate counting, we use the local lemma again, to find an assignment so that every hyperedge is satisfied by the first $\frac{k}{14}$ vertices. Then we compute the marginal probability of this assignment by fixing vertices one by one.

- For sampling, we use the marginal to colour vertices, similar to the coupling process. With high probability, the remaining connected components are of size $O(\log n)$.

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Concluding remarks

Open problems

- What is the correct threshold for hypergraph colouring?
  - Is it $q \asymp \Delta^{\frac{2}{k}}$?
- What about **NP**-hardness of sampling hypergraph colourings?
- Does this method work for general LLL?
- What is the relationship between this method and traditional ones (Markov chains, spatial mixing, etc.)?

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A professor is one who can speak on any subject for precisely fifty minutes.

— Norbert Wiener

Thank you!

arXiv:1711.03396