Random Cluster Dynamics for the Ising model is Rapidly Mixing

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The model and its dynamics

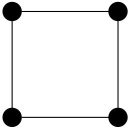
The random cluster model [Fortuin, Kasteleyn 1969]

Parameters $0 \leqslant p \leqslant 1$ (edge weight), $q \geqslant 0$ (cluster weight).

Given graph $G = (V, E)$, the measure on subgraph $r \subseteq E$ is defined as

$$\pi_{RC}(r) \propto p^{|r|}(1-p)^{|E \setminus r|} q^{\kappa(r)},$$

where $\kappa(r)$ is the number of connected components in (V, r).



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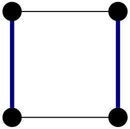

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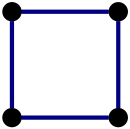

$p^4 q$

The partition function (normalizing factor):

$$Z_{RC}(p,q) = \sum_{r \subseteq E} p^{|r|}(1-p)^{|E \setminus r|} q^{\kappa(r)}.$$

Equivalent to the Tutte polynomial $Z_{Tutte}(x,y)$:

$$q = (x-1)(y-1) \qquad\qquad p = 1 - \frac{1}{y}$$

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The motivation is to unify:

- Ising model
- Potts model
- Bond percolation
- Electrical network

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Glauber dynamics

Glauber dynamics (single edge update) P_{RC} (Metropolis):

Current state $x \subseteq E$

1. With prob. $1/2$ do nothing. (Lazy)
2. Otherwise, choose an edge e u.a.r.
3. Move to $y = x \oplus \{e\}$ with prob. $\min\left\{1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)}\right\}$.

Detailed balance:

$$\pi(x)P(x, y) = \pi(y)P(y, x) = \min\{\pi(x), \pi(y)\}$$

Glauber dynamics

Glauber dynamics (single edge update) P_{RC} (Metropolis):

$$P_{RC}(x,y) = \begin{cases} \frac{1}{2m}\min\left\{1, \frac{\pi_{RC}(y)}{\pi_{RC}(x)}\right\} & \text{if } |x \oplus y| = 1; \\ 1 - \frac{1}{2m}\sum_{e\in E}\min\left\{1, \frac{\pi_{RC}(x\oplus\{e\})}{\pi_{RC}(x)}\right\} & \text{if } x = y; \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in the mixing time $\tau_\epsilon(P_{RC})$:

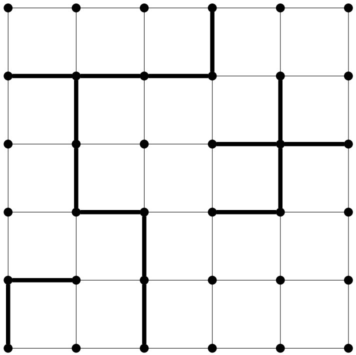
$$\tau_\epsilon(P_{RC}) = \min\left\{t : \|P_{RC}^t(x_0,\cdot) - \pi\|_{TV} \leqslant \epsilon\right\}.$$

A simple example

Let $p < 1/2$.

$$\min\left\{1, \frac{\pi_{RC}(x \cup \{e\})}{\pi_{RC}(x)}\right\}$$

$$= \begin{cases} \frac{p}{1-p} & \text{if } e \text{ is not a cut edge} \\ \frac{p}{q(1-p)} & \text{if } e \text{ is a cut edge} \end{cases}$$



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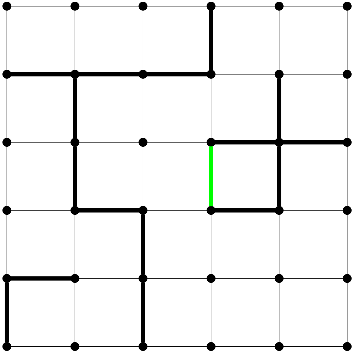

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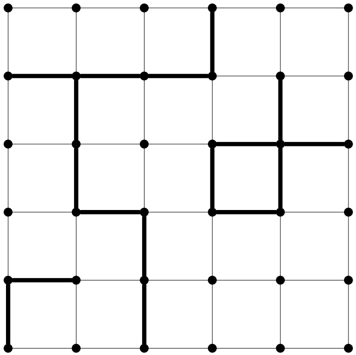

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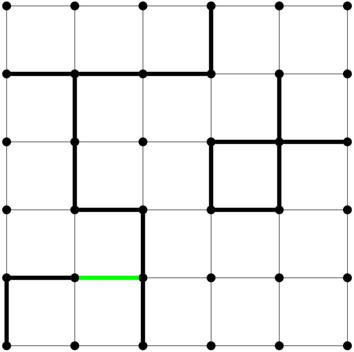

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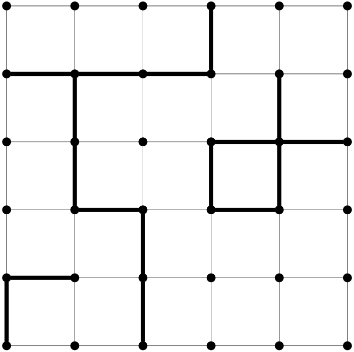

Brief History

Studied extensively for special graphs,
such as the complete graph (mean-field) and the lattice $\mathbb{Z}^2$.

- Mean-field: [Gore, Jerrum 1999]
  [Blanca, Sinclair 2015]
- $\mathbb{Z}^2$: [Borgs et al. 1999]
  [Blanca, Sinclair 2016]
  [Gheissari, Lubetzky 2016]

$q > 2$: Slow mixing for the complete graph.
$0 \leqslant q \leqslant 2$: No known fast mixing bound for general graphs.

Main theorem

Theorem

For the random cluster model with parameters $0 < p < 1$ and $q = 2$,

$$\tau_\epsilon(P_{RC}) \leqslant 10 n^4 m^2 (\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1}).$$

- For $q > 2$, there exists p such that P_{RC} is slow mixing on complete graphs. [Gore, Jerrum 1999] [Blanca, Sinclair 2015]
- For $q > 2$ and $0 < p < 1$, it is #BIS-hard to approximate $Z_{RC}(p, q)$. [Goldberg, Jerrum 2012]
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Swandsen-Wang algorithm

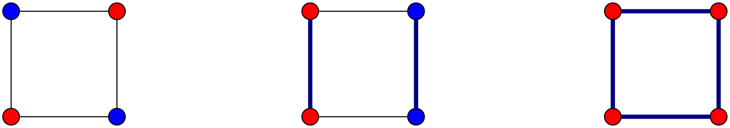
Ferromagnetic Ising model [Ising, Lenz 1925]

Parameter $\beta > 1$.

A configuration $\sigma : V \to \{+, -\}$.

$$\pi_{Ising}(\sigma) \propto \beta^{mono(\sigma)} = \beta^{m-cut(\sigma)}$$

Partition function $Z_{Ising}(\beta) = \sum_{\sigma} \beta^{mono(\sigma)}$



β^0 β^2 β^4

Equivalence at $q = 2$

Let $\beta = \frac{1}{1-p}$.

$$Z_{Ising}(\beta) = \beta^{|E|} Z_{RC}(p, 2)$$

Swendsen-Wang algorithm [Swendsen, Wang 1987]

A global Markov chain to sample Ising configurations.

Current configuration σ

1. Mark all monochromatic edges under σ as M
2. Remove each edge in M with probability β^{-1} (Recall $\beta^{-1} = 1 - p$)
3. Assign a random spin to each component of (V, M)

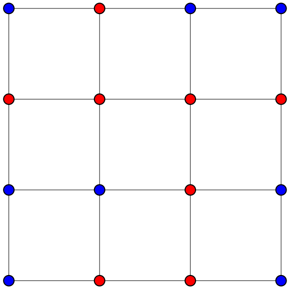
Practically very fast for the Ising model, but difficult to analyze.

Conjectured to be rapidly mixing for all graphs.

(Open problem since 90s.)

Another simple example

1. Activate mono edges
2. Re-randomize mono edges
3. Color components



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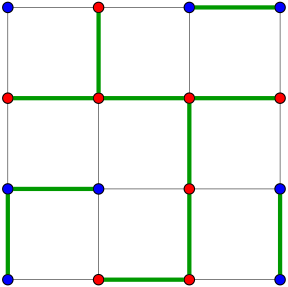

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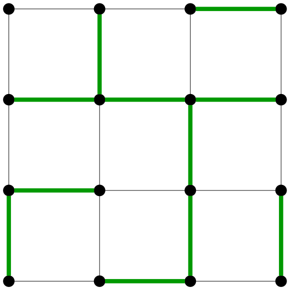

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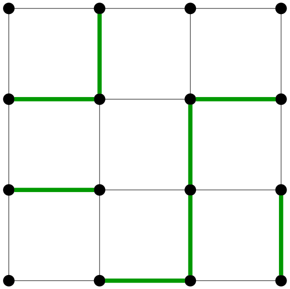

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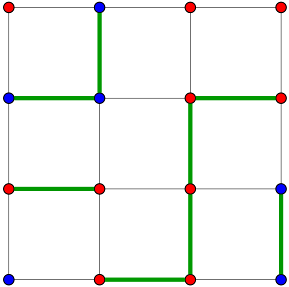

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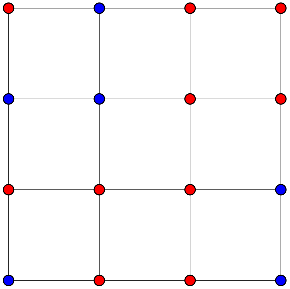

Previous Results

- Swendsen-Wang algorithm on the complete graph:

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Combine with our theorem:

the Swendsen-Wang algorithm is rapidly mixing at $q = 2$,

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Even subgraphs

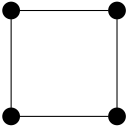
Another equivalent formulations at $q = 2$

Even subgraphs

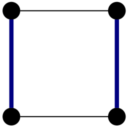
Let $r \subseteq E$ such that every vertex in (V, r) has an even degree.

$$\pi_{even}(r) \propto p^{|r|}(1-p)^{|E \setminus r|}$$

Partition function $Z_{even}(p)$



$(1-p)^4$



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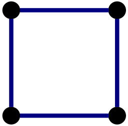

p^4

Equivalence at $q = 2$

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$$Z_{Ising}(\beta) \;=\; \beta^{|E|} Z_{RC}(p, 2) \;=\; 2^{|V|} \beta^{|E|} Z_{even}\left(\frac{p}{2}\right)$$

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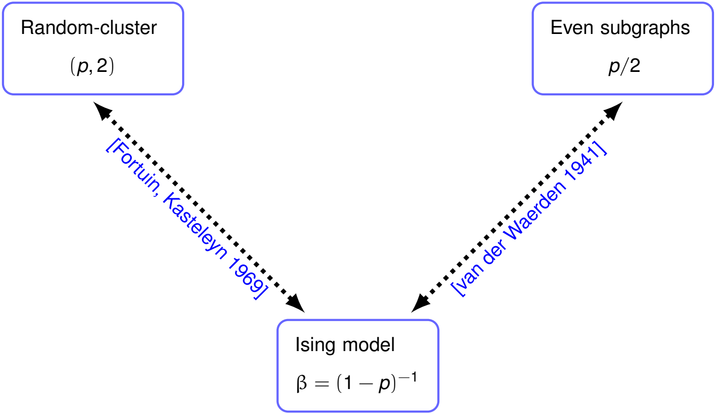

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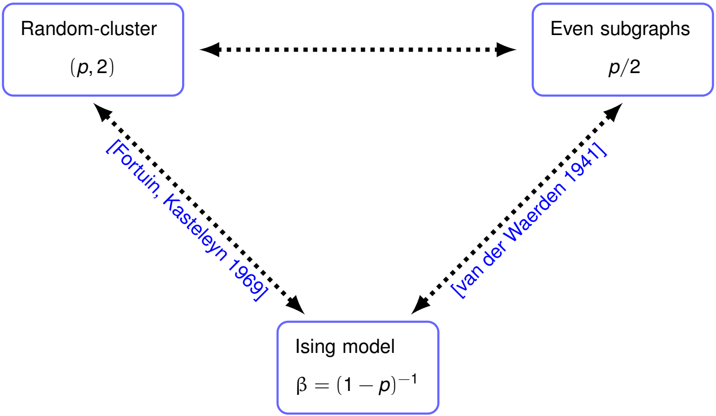

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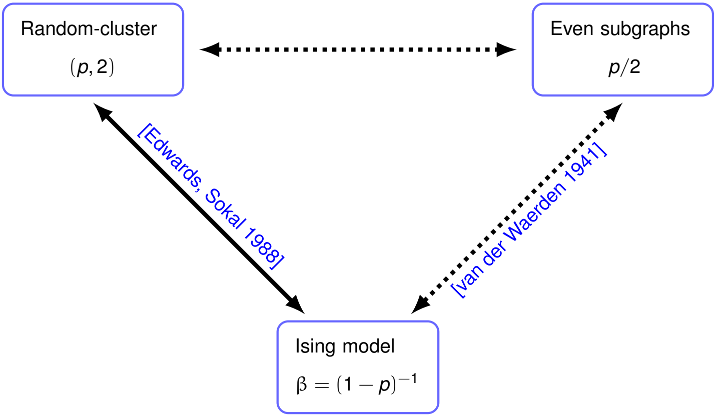

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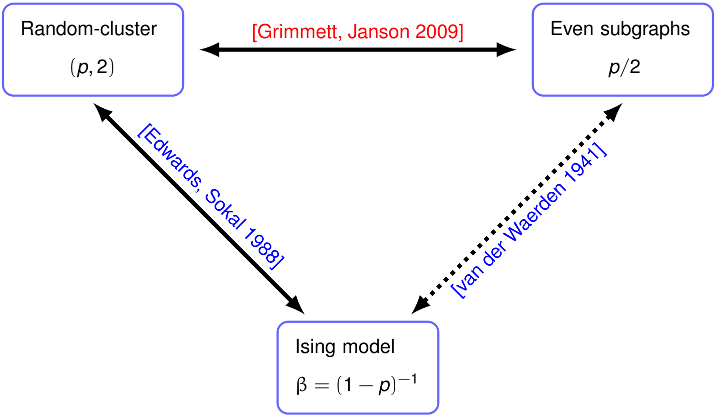

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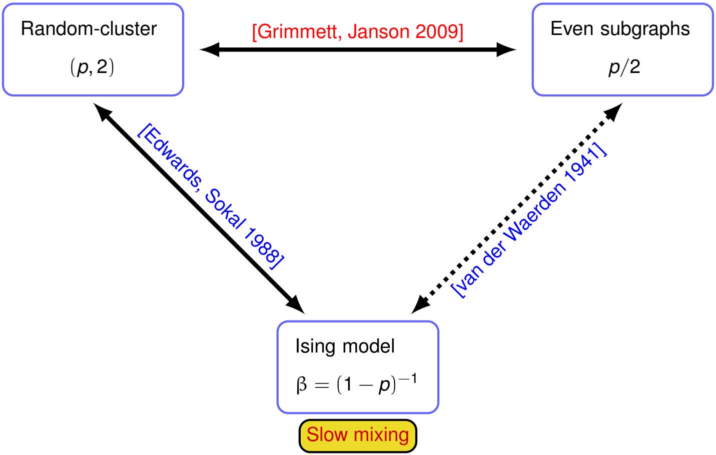

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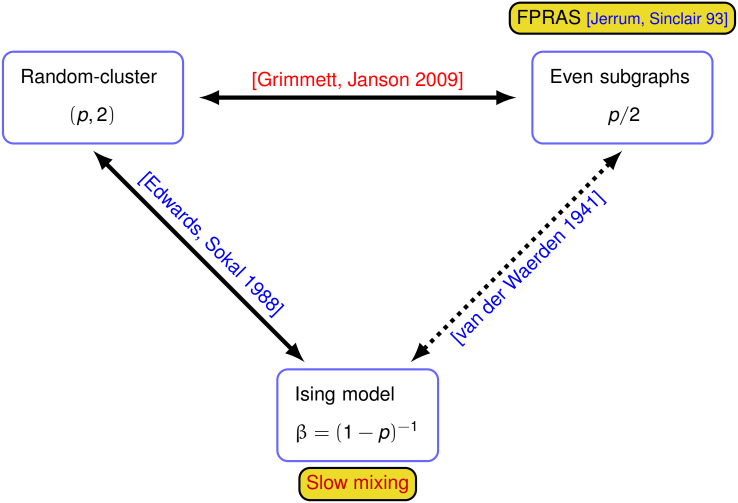

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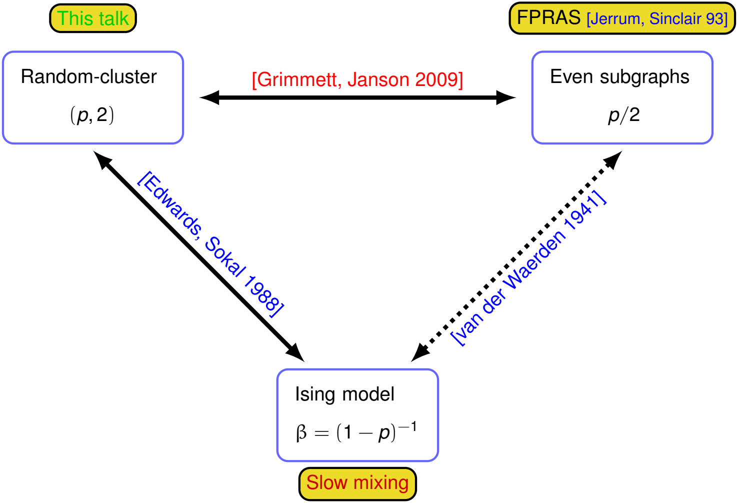

Grimmett-Janson coupling

Given a graph G, draw a random even subgraph $S \subseteq E$ with $p \leqslant \frac{1}{2}$:

$$\Pr(S = s) = \pi_{even}(s).$$

Then we add every edge $e \notin S$ with probability $p' = \frac{p}{1-p}$.

Call this subgraph R.

Theorem (Grimmett, Janson 2009)

$$\Pr(R = r) = \pi_{RC;2p,2}(r).$$

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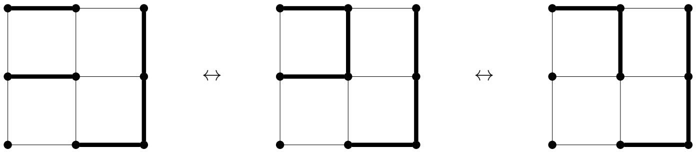
The Proof

Bound the mixing time

- A Markov chain is a random walk on its state space (exponentially large).



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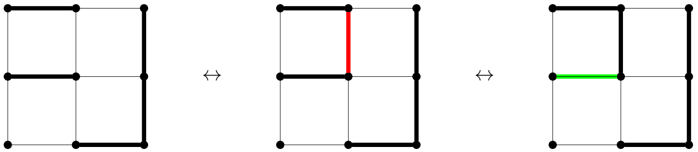

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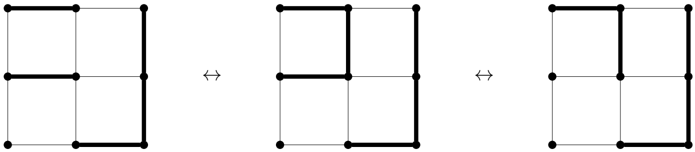

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  - Two configurations are adjacent if they differ by exactly one edge.

- Rapidly mixing $\Leftrightarrow$ The state space is very well connected.

Congestion and canonical paths

- Construct a set Γ of canonical paths γ_{xy} for all pairs of states (x, y).

  The key quantity of Γ is its congestion:

$$\rho(\Gamma) := \max_{\substack{(z,z')\in\Omega^2 \\ P(z,z')>0}} \frac{L}{\pi(z)P(z,z')} \sum_{\substack{x,y\in\Omega^2 \\ \gamma_{xy}\ni(z,z')}} w(\gamma_{xy}),$$

  where

$$w(\gamma_{xy}) = \pi(x)\pi(y).$$

Theorem (Sinclair 1992)

$$\tau_\varepsilon(P) \leqslant \rho(\Gamma)(\ln \pi(x_0)^{-1} + \ln \varepsilon^{-1}).$$

Alternative view of canonical paths

Fix $\Gamma = \{\gamma_{xy}\}$ and an integer $k \leqslant L$.

1. Draw the initial and final states I and F independently according to $\pi(\cdot)$.
2. A random path $\gamma_{IF} \in \Gamma$.

$$\mu(\gamma_{IF}) = w(\gamma_{IF}) = \pi(I)\pi(F)$$

3. Let Z_k be the kth state of γ_{IF}.

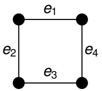
   (Assume all paths in Γ have the same length L.)

The congestion $\rho(\Gamma)$ is polynomial related with $\max_k \frac{\Pr(Z_k=z)}{\pi(z)}$.

Alternative view in action

Let $q = 1$. Then $\pi_{RC}(\cdot)$ is a product measure.

G:



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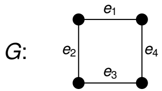

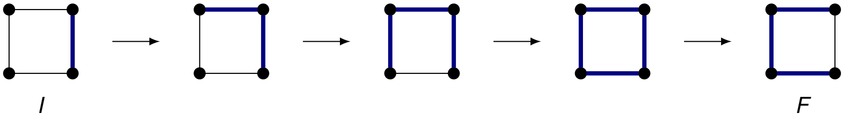

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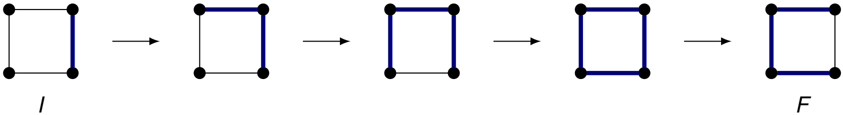

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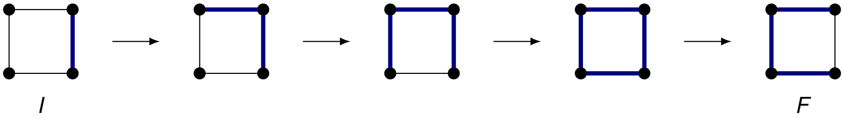

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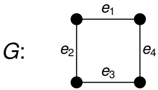

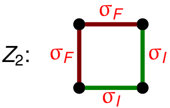

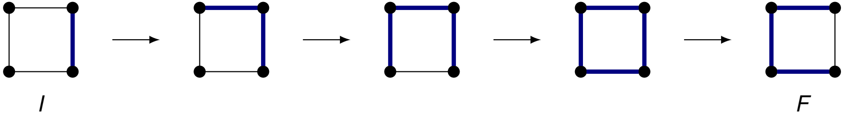

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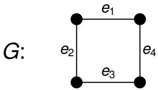
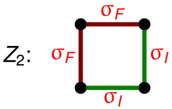
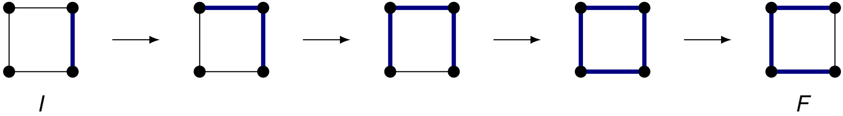






$$\frac{\Pr(Z_k = z)}{\pi(z)} = 1$$

From paths to flows

Instead of one path from x to y, we can have a random path from x to y.

Flow Γ is a collection of paths equipped with weights $w(\cdot)$ such that

$$\sum_{\gamma \text{ is from } x \text{ to } y} w(\gamma) = \pi(x)\pi(y).$$

Z_k is defined similarly.

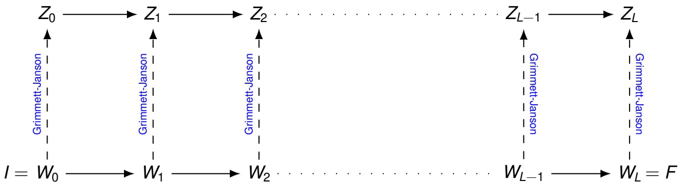
1. Random initial and final states I and F
2. A random path γ from I to F according to $w(\cdot)$.
3. Z_k is the kth state of γ.

We will look at $\frac{\Pr(Z_k=z)}{\pi(z)}$.

Lifting canonical paths

In an ideal world ...

- Suppose we have canonical paths Γ_{even} for even subgraphs with low congestion. (similar to [Jerrum, Sinclair 93])
- Then use Grimmett-Janson to lift Γ_{even} to a flow for random cluster.



- $w(\zeta) = w(\gamma)\Pr(\gamma \to \zeta)$

Ideal lifting

If W_k deviates from $\pi_{even}(\cdot)$ by at most polynomial, then so does Z_k from $\pi_{RC}(\cdot)$.

$$\frac{Pr(W_k = w)}{\pi_{even}(w)} \leqslant n^{O(1)} \rho(\Gamma)$$

$$\begin{aligned} Pr(Z_k = z) &= \sum_{w \subseteq z,\, w \text{ even}} Pr(W_k = w) \left(\frac{p}{1-p}\right)^{|z \setminus w|} \left(\frac{1-2p}{1-p}\right)^{|E \setminus z|} \\ &\leqslant n^{O(1)} \rho(\Gamma) \sum_{w \subseteq z,\, w \text{ even}} \pi_{even}(w) \left(\frac{p}{1-p}\right)^{|z \setminus w|} \left(\frac{1-2p}{1-p}\right)^{|E \setminus z|} \\ &= n^{O(1)} \rho(\Gamma) \pi_{RC}(z) \qquad \text{(by GJ)} \end{aligned}$$

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In the real world . . .

Two issues:

1. We do not have good canonical paths for even subgraphs —
   Jerrum-Sinclair chain moves among all subgraphs!

2. Grimmett-Janson adds indepdendent edges —
   Z_i and Z_{i+1} are not adjacent states!
   They may differ by a lot of edges.

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Patch 1

Issue 1: need canonical paths for even subgraphs.

- Construct paths $\Gamma_{even} = \{\gamma_{xy}\}$ where x and y are both even subgraphs.
  - $x \oplus y$ is also even.

    $x \oplus y$ can be covered by edge-disjoint cycles.
  - Pick a canonical ordering of edges. Unwind each cycle:

    $W_0 = x$, $W_i = W_{i-1} \oplus e_i$
  - Enlarge the state space to all even and near-even subgraphs.

    Every path is in the augmented space.

- Γ_{even} has low congestion — same reason as [Jerrum, Sinclair 1993].

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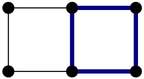
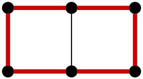
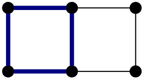


$x = Z_0$



$x \oplus y$



$y = Z_6$

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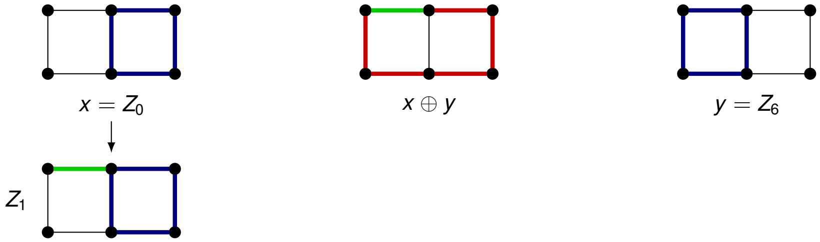

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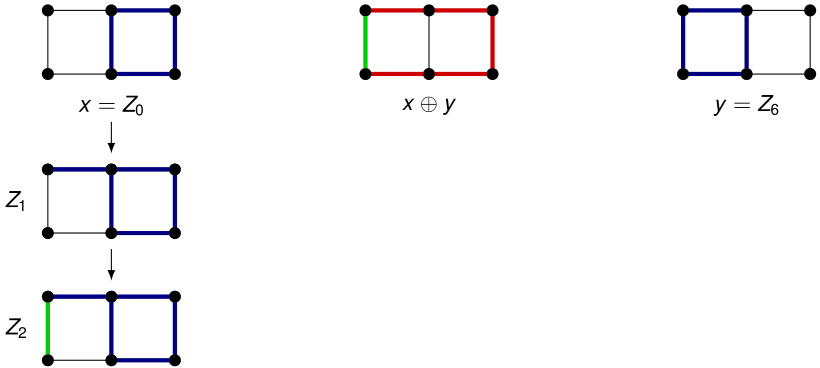

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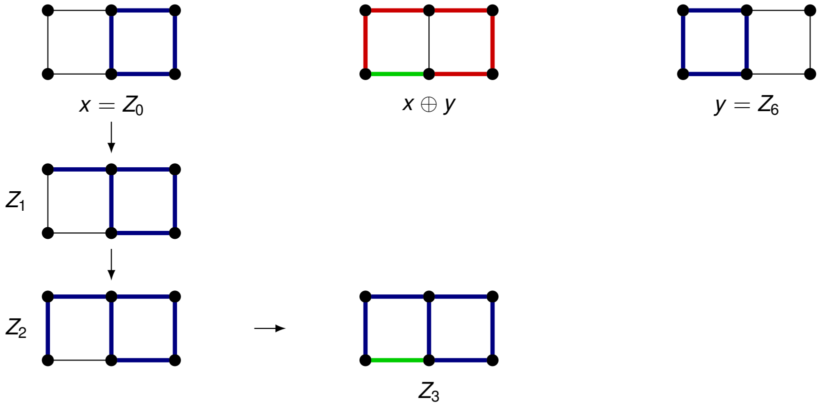

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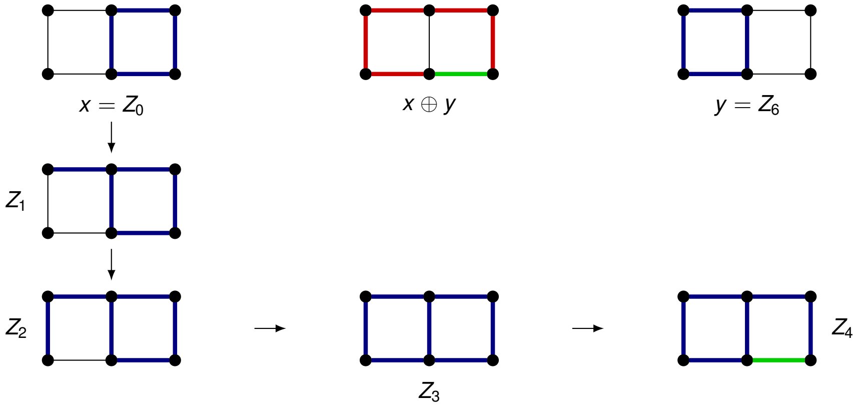

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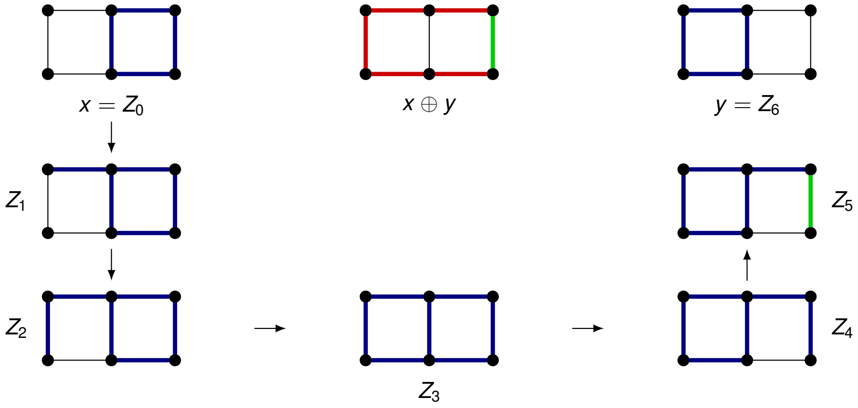

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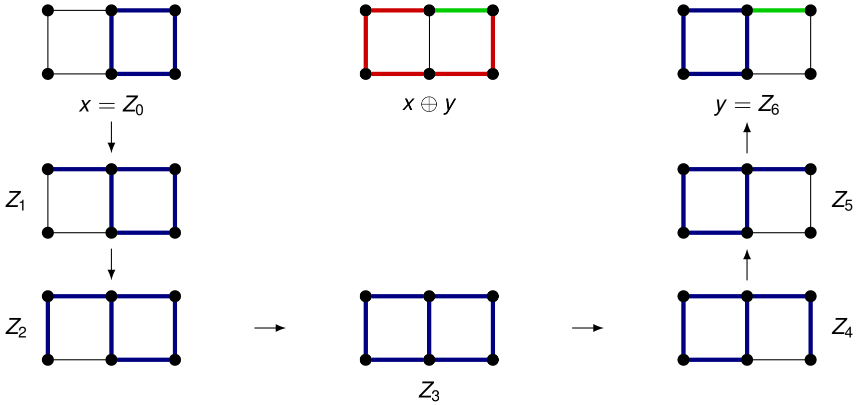

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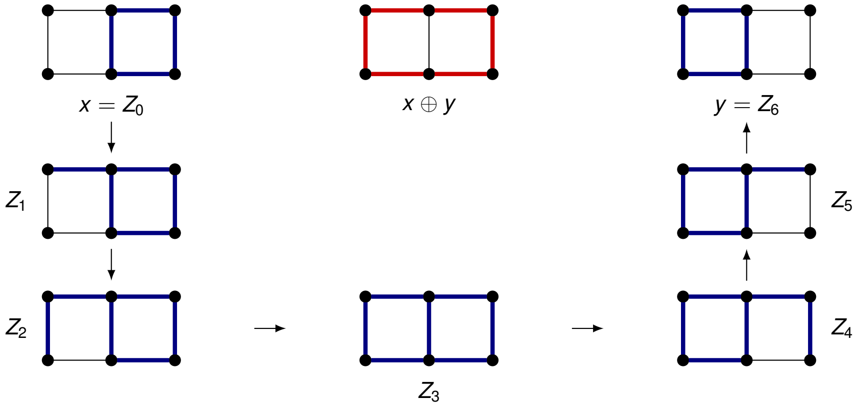

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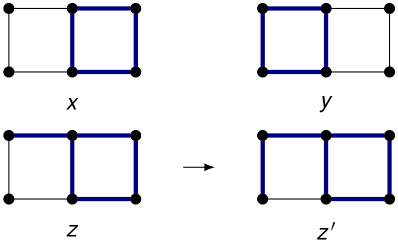

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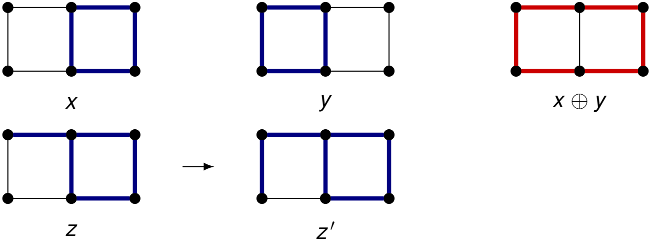

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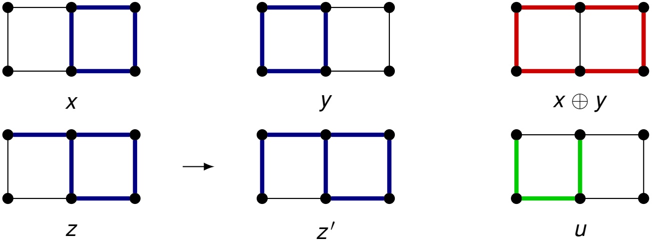

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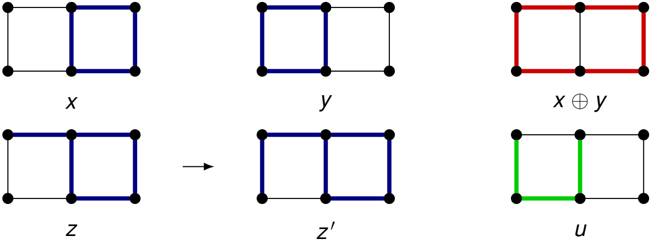

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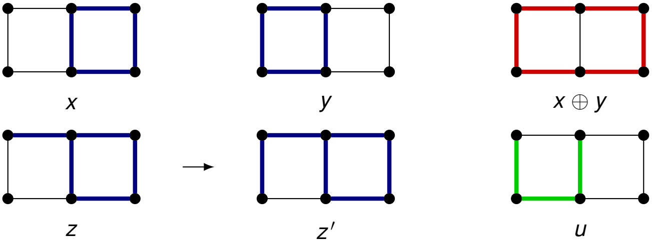

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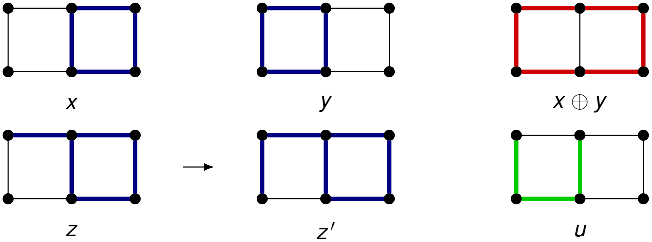

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One final problem for issue 1:

- W_0 and W_L are both even,
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A generalization of Grimmett-Janson:

- Give each near-even subgraph a penalty of $1/n^2$.
- Add independent edges with prob. $\frac{p}{1-p}$ as before.
  Call the resulting measure $\widehat{\pi}(\cdot)$.
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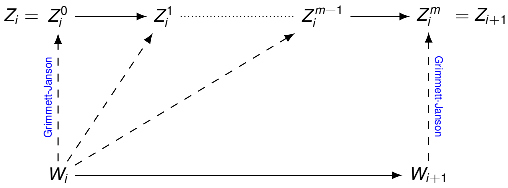
Patch 2

Issue 2: Z_i and Z_{i+1} differ by more than 1 edge.

- An easy fix: insert intermediate states to change edges one by one in $Z_i \oplus Z_{i+1}$, which has a product measure on $E \backslash (W_i \cup W_{i+1})$.



- The distribution of Z_i^j is the same as that of Z_i $(j < m)$.
- Total length is mL.

Better patch 2

Issue 2: Z_i and Z_{i+1} differ by more than 1 edge.

- Lift W_{i+1} to Z_{i+1} conditional on Z_i such that Z_{i+1} and Z_i are adjacent and the marginal of Z_{i+1} is correct.
- The marginal distributions of Z_0 and Z_L are correct, but their joint distribution is not — Z_0 and Z_L are correlated.
- Append a tail on the path after Z_L to re-randomize edges that are not in W_L. This removes the correlation.
- Total length is at most $L + m$.

Putting everything together

Z_0 Z_{L+m}

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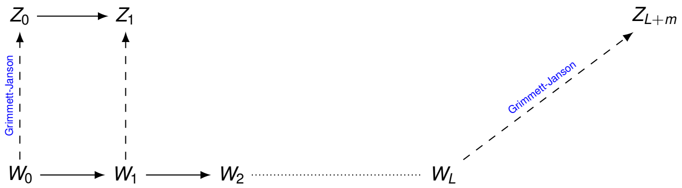

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- $W_2 = W_1 \setminus \{e'\} \quad \Rightarrow \quad Z_2 = \begin{cases} Z_1 & \text{prob. } p' \\ Z_1 \setminus \{e'\} & \text{prob. } 1 - p' \end{cases}$

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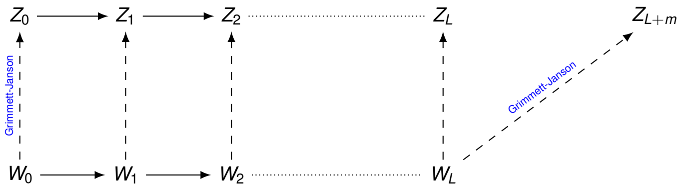

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Future directions

Tutte polynomial [Goldberg, Jerrum 08,12,14]

$q = (x-1)(y-1)$

Tractable

FPRAS

NP-hard

(most #**P**-hard)

#PM-equivalent

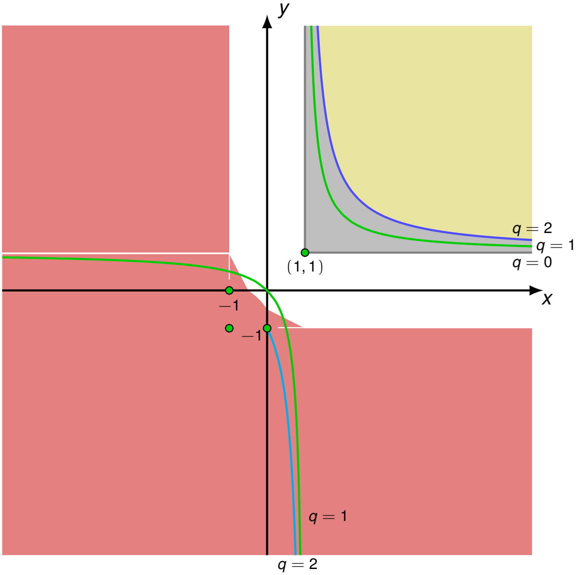
#BIS-hard

Open:

All white

$0 \leqslant q < 1$

$1 < q < 2$



Recap

Theorem

$$\text{At } q = 2,\ \tau_\epsilon(P_{RC}) \leqslant 10n^4 m^2 (\ln \pi_{RC}(x_0)^{-1} + \ln \epsilon^{-1}).$$

- $q = 2$ tighter mixing time bound?
- $1 < q < 2$ (monotone) fast mixing?
- $0 \leqslant q < 1$ (e.g. Tutte(2,1) = #Forests) fast mixing???

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Thank You!

Paper available: `arxiv.org/abs/1605.00139`