The Complexity of Planar Boolean #CSP with Complex Weights

Heng Guo
(joint work with Tyson Williams)

University of Wisconsin-Madison

Riga, Latvia
July 8th 2013

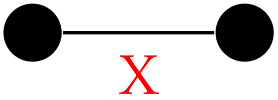
Definition

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex in the set.

#VertexCover

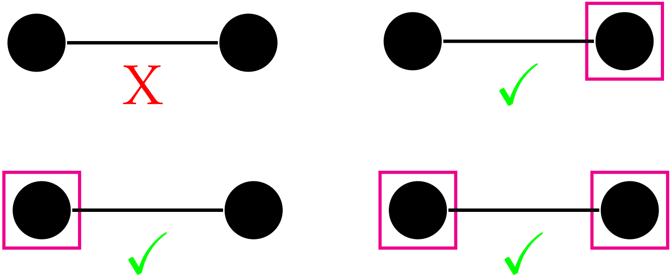
Definition

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex in the set.



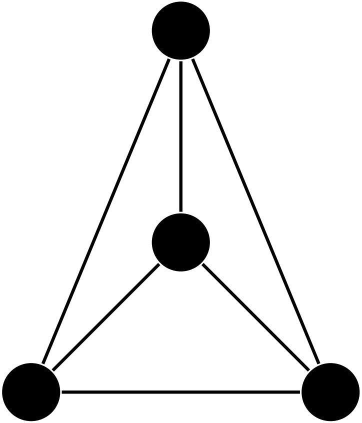
Definition

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex in the set.



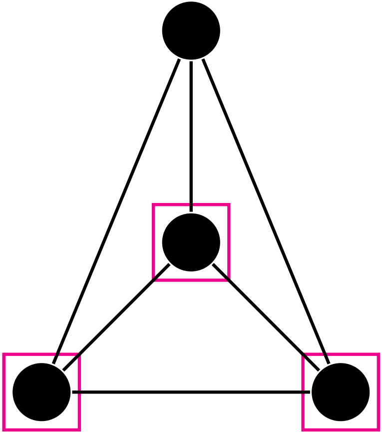
Systematic Approach to #VertexCover

- $G = (V, E)$



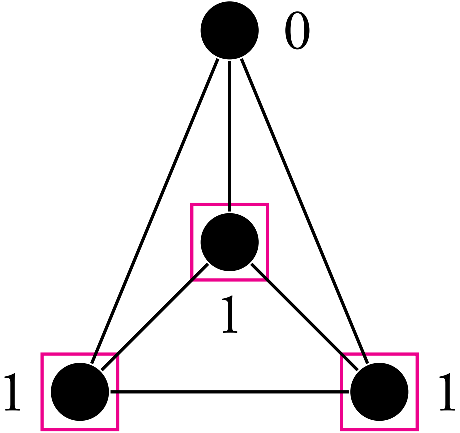
Systematic Approach to #VertexCover

- $G = (V, E)$



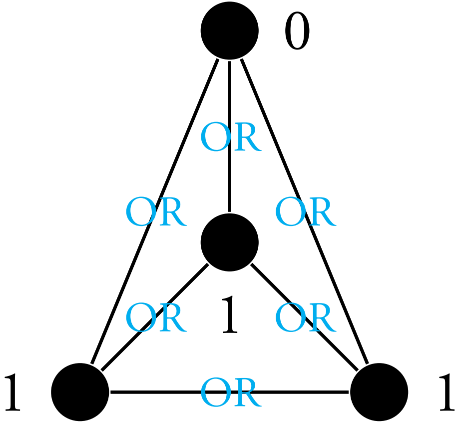
Systematic Approach to #VertexCover

- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



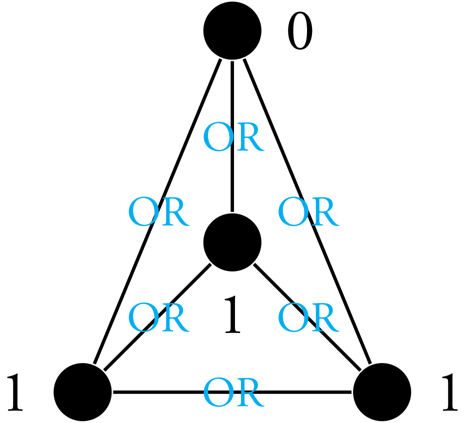
Systematic Approach to #VertexCover

- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



Systematic Approach to #VertexCover

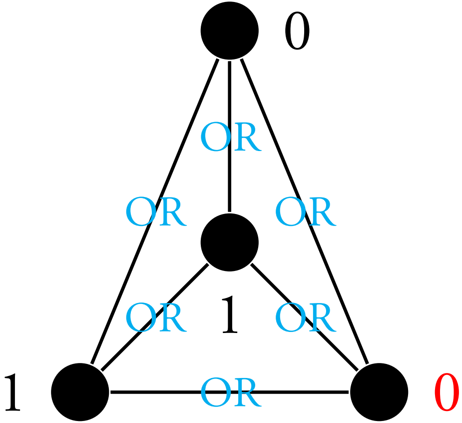
- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



$$\prod_{(u,v)\in E} \text{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

Systematic Approach to #VertexCover

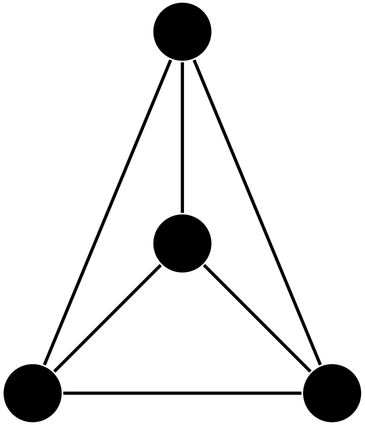
- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$$

Systematic Approach to #VertexCover

- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



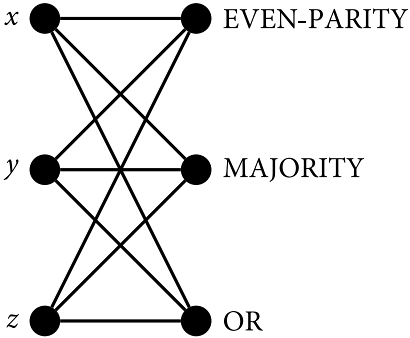
$$\#\textsc{VertexCover}(G) = \sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v))$$

Constraint Graph

$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$

Constraint Graph

$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$



Constraint Satisfaction Problems

#CSP($\mathcal{F}$)

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

where $N(c)$ are the neighbors of c.

Constraint Satisfaction Problems

#CSP($\mathcal{F}$)

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

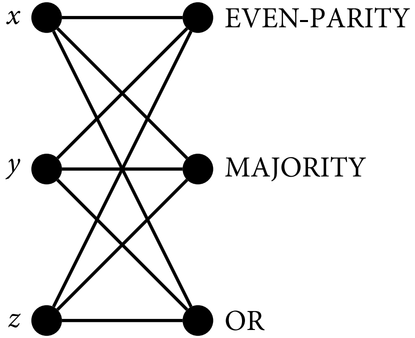
  where $N(c)$ are the neighbors of c.
- In this talk we consider the case where the constraint graph is planar, denoted Pl-#CSP($\mathcal{F}$).

Planar Constraint Graph

$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$

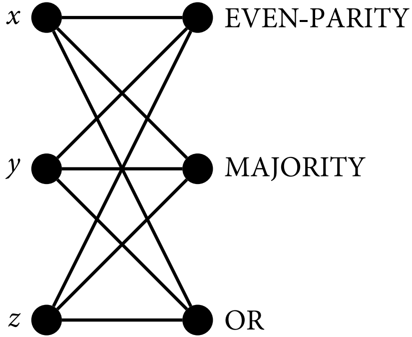
Planar Constraint Graph

$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$



Planar Constraint Graph

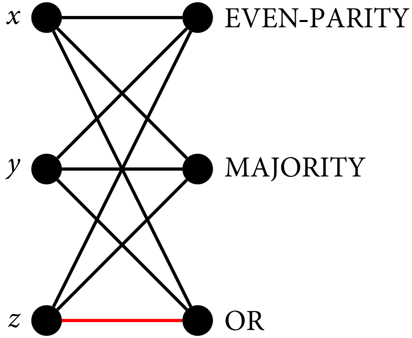
$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$



NOT planar, so **NOT** an instance of
Pl-#CSP($\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$)

Planar Constraint Graph

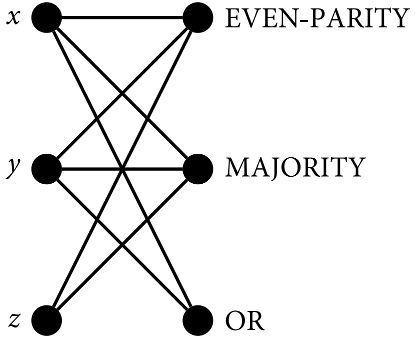
$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y, z)$$



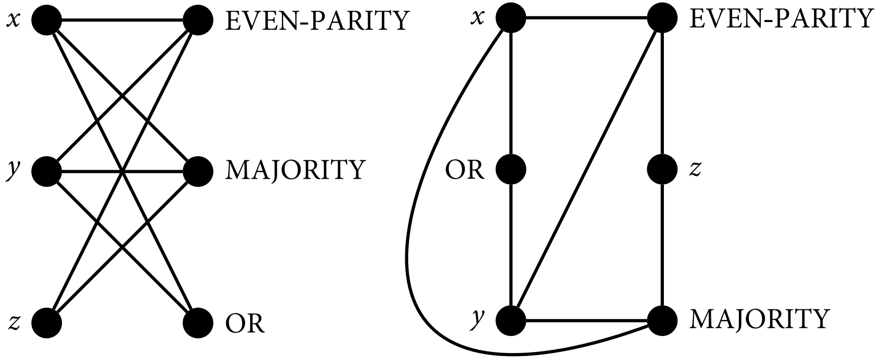
NOT planar, so **NOT** an instance of
Pl-#CSP($\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$)

Planar Constraint Graph

$$\text{EVEN-PARITY}(x, y, z) \wedge \text{MAJORITY}(x, y, z) \wedge \text{OR}(x, y)$$



EVEN-PARITY$(x, y, z) \wedge$ MAJORITY$(x, y, z) \wedge$ OR(x, y)



VALID instance of Pl-#CSP$(\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_2\})$

$\#\mathrm{CSP}(\mathcal{F})$ in Holant Framework

$\#\mathrm{CSP}(\mathcal{F})$

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c\left(\sigma \mid_{N(c)}\right),$$

where $N(c)$ are the neighbors of c.

#CSP($\mathcal{F}$) in Holant Framework

#CSP($\mathcal{F}$)

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma:V\to\{0,1\}} \prod_{c\in C} f_c\left(\sigma\mid_{N(c)}\right),$$

where $N(c)$ are the neighbors of c.

Holant($\mathcal{F}$)

- On input graph $G = (V, E)$, compute

$$\sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v\left(\sigma\mid_{E(v)}\right),$$

where $E(v)$ are the incident edges of v.

#CSP($\mathcal{F}$) in Holant Framework

#CSP($\mathcal{F}$)

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma : V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

where $N(c)$ are the neighbors of c.

Holant($\mathcal{F}$)

- On input graph $G = (V, E)$, compute

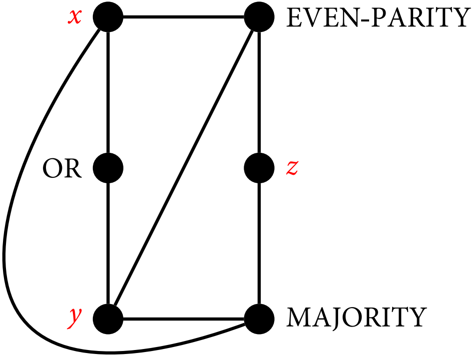
$$\sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)} \right),$$

where $E(v)$ are the incident edges of v.

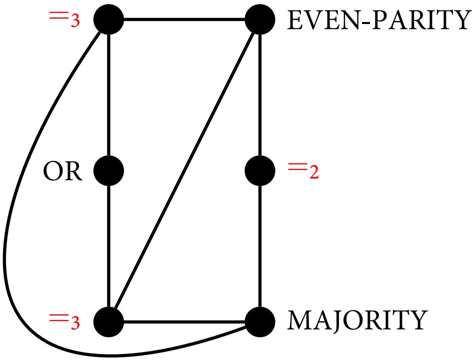
$$\#\mathrm{CSP}(\mathcal{F}) \equiv_T \mathrm{Holant}(\mathcal{EQ} \cup \mathcal{F}),$$

where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.

Example



Example



Symmetric signatures

Symmetric Signatures: value only depends on the Hamming weight of the inputs.

$$\begin{aligned} \text{OR}_2 &= [0, 1, 1] \\ \text{AND}_3 &= [0, 0, 0, 1] \\ \text{EVEN-PARITY}_4 &= [1, 0, 1, 0, 1] \\ \text{MAJORITY}_5 &= [0, 0, 0, 1, 1, 1] \\ (=_6) = \text{EQUALITY}_6 &= [1, 0, 0, 0, 0, 0, 1] \end{aligned}$$

Holographic transformation

- The action of a 2-by-2 non-singular matrix T on a signature f of arity n is $T^{\otimes n} f$. We use $T\mathcal{F}$ to denote that T acts upon every element of $\mathcal{F}$.

Holographic transformation

- The action of a 2-by-2 non-singular matrix T on a signature f of arity n is $T^{\otimes n} f$. We use $T\mathcal{F}$ to denote that T acts upon every element of $\mathcal{F}$.
- Holographic transformation: If H is orthogonal, then Valiant's Holant theorem implies that $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(H\mathcal{F})$.

Holographic transformation

- The action of a 2-by-2 non-singular matrix T on a signature f of arity n is $T^{\otimes n} f$. We use $T\mathcal{F}$ to denote that T acts upon every element of $\mathcal{F}$.
- Holographic transformation: If H is orthogonal, then Valiant's Holant theorem implies that $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(H\mathcal{F})$.
- Example: Let $H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H_2^{\otimes n}(=_n) = \text{EVEN-PARITY}_n$$

Holographic transformation

- The action of a 2-by-2 non-singular matrix T on a signature f of arity n is $T^{\otimes n} f$. We use $T\mathcal{F}$ to denote that T acts upon every element of $\mathcal{F}$.
- Holographic transformation: If H is orthogonal, then Valiant's Holant theorem implies that $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(H\mathcal{F})$.
- Example: Let $H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$H_2^{\otimes n}(=_n) = \text{EVEN-PARITY}_n$$

Let $H_2\mathcal{F} = \widehat{\mathcal{F}}$

Holographic transformation

- The action of a 2-by-2 non-singular matrix T on a signature f of arity n is $T^{\otimes n} f$. We use $T\mathcal{F}$ to denote that T acts upon every element of $\mathcal{F}$.
- Holographic transformation: If H is orthogonal, then Valiant's Holant theorem implies that $\text{Holant}(\mathcal{F}) \equiv_T \text{Holant}(H\mathcal{F})$.
- Example: Let $H_2 = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$

$$H_2{}^{\otimes n}(=_n) = \text{EVEN-PARITY}_n$$

Let $H_2\mathcal{F} = \widehat{\mathcal{F}}$
Note: $H_2\widehat{\mathcal{F}} = \mathcal{F}$ since $H_2\widehat{\mathcal{F}} = H_2H_2\mathcal{F} = \mathcal{F}$

Some Signature Sets

Affine signatures $\mathscr{A}$:

1. $[1, 0, \ldots, 0, \pm 1]$
2. $[1, 0, \ldots, 0, \pm i]$
3. $[1, 0, 1, 0, \ldots, 0 \text{ or } 1]$
4. $[1, -i, 1, -i, \ldots, (-i) \text{ or } 1]$
5. $[0, 1, 0, 1, \ldots, 0 \text{ or } 1]$
6. $[1, i, 1, i, \ldots, i \text{ or } 1]$
7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$
8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1 \text{ or } (-1)]$
9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$
10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1 \text{ or } (-1)]$

Product-type signatures $\mathscr{P}$:

1. $[0, x, 0]$
2. $[y, 0, \ldots, 0, z]$ (includes all unary signatures)

Some Signature Sets

Matchgate signatures $\mathscr{M}$:

1. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
2. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$
3. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
4. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$

Some Signature Sets

Matchgate signatures $\mathscr{M}$:

1. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
2. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$
3. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
4. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$

Example

$$\widehat{\mathcal{EQ}} = \{\text{EVEN-PARITY}_n \mid n \in \mathbb{Z}^+\}$$

Previous Work: Planar Dichotomy Theorems

[Cai, Lu, Xia 10]

- Dichotomy for Pl-#CSP($\mathcal{F}$) with **real** weights

Previous Work: Planar Dichotomy Theorems

[Cai, Lu, Xia 10]

- Dichotomy for Pl-#CSP($\mathcal{F}$) with **real** weights
- Dichotomy for Pl-Holant(f) for arity 3 signature with **complex** weights

Previous Work: Planar Dichotomy Theorems

[Cai, Lu, Xia 10]

- Dichotomy for Pl-#CSP($\mathcal{F}$) with **real** weights
- Dichotomy for Pl-Holant(f) for arity 3 signature with **complex** weights

[Cai, Kowalczyk 10]

- Dichotomy for Pl-#CSP($[a, b, c]$) with **complex** weights

Main Result

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, *or* $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, *in which case the problem is in* P.

Main Result

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, *or* $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, *in which case the problem is in* P.

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard unless* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, *or* $\mathcal{F} \subseteq \mathscr{M}$, *in which case the problem is in* P.

Secondary Result

Theorem

If f is a non-degenerate, symmetric, complex-valued signature of arity 4 in Boolean variables, then Pl-Holant(f) is #P-hard unless f is

- *$\mathscr{A}$-transformable,*
- *$\mathscr{P}$-transformable,*
- *vanishing, or*
- *$\mathscr{M}$-transformable,*

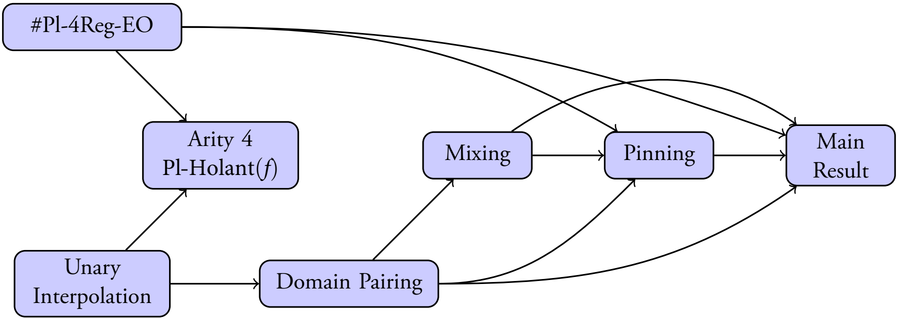
in which case the problem is in P.

Definition ($\mathcal{F}$-transformable)

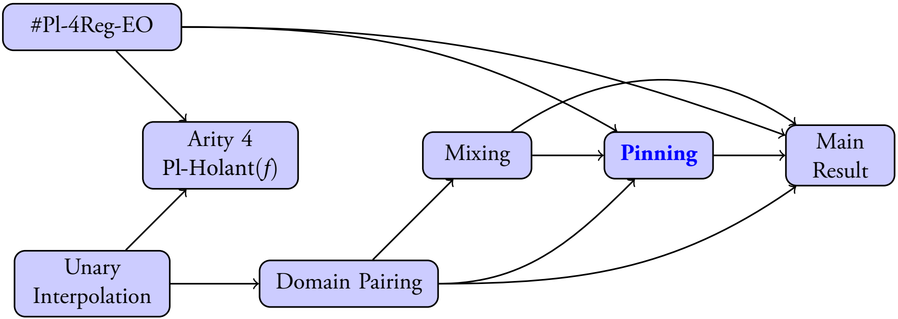
A signature f is $\mathcal{F}$-transformable if there exists $T \in \mathbb{C}^{2\times 2}$ such that

- $f \in T\mathcal{F}$ and
- $=_2 T^{\otimes 2} \in \mathcal{F}$.

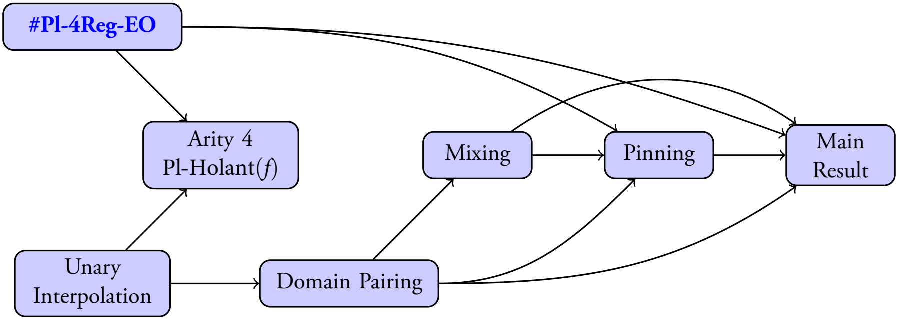
Proof Outline: Dependency Graph



Proof Outline: Dependency Graph



Proof Outline: Dependency Graph



Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe Jerrum, Thurley 10]
- [Cai, Chen, Lu 10]

#CSP

- [Bulatov, Dalmau 07]
- [Dyer, Goldberg, Jerrum 09]
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09]
- [Cai, Lu, Xia 10]
- [Huang, Lu 12]

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe Jerrum, Thurley 10]
- [Cai, Chen, Lu 10]

#CSP

- [Bulatov, Dalmau 07]
- [Dyer, Goldberg, Jerrum 09]
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09]
- [Cai, Lu, Xia 10]
- [Huang, Lu 12]

Lemma (Dyer, Goldberg, Jerrum 09)

For complex weights, $\#\text{CSP}(\mathcal{F} \cup \{[1,0],[0,1]\}) \leq_T \#\text{CSP}(\mathcal{F})$.

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe Jerrum, Thurley 10]
- [Cai, Chen, Lu 10]

#CSP

- [Bulatov, Dalmau 07]
- [Dyer, Goldberg, Jerrum 09]
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09]
- [Cai, Lu, Xia 10]
- [Huang, Lu 12]

Lemma (Dyer, Goldberg, Jerrum 09)

For complex weights, $\#\mathrm{CSP}(\mathcal{F} \cup \{[1,0],[0,1]\}) \leq_T \#\mathrm{CSP}(\mathcal{F})$.

Pl-#CSP$(\widehat{\mathscr{M}} \cup \{[1,0],[0,1]\})$ #P-hard but Pl-#CSP$(\widehat{\mathscr{M}})$ tractable

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe Jerrum, Thurley 10]
- [Cai, Chen, Lu 10]

#CSP

- [Bulatov, Dalmau 07]
- [Dyer, Goldberg, Jerrum 09]
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09]
- [Cai, Lu, Xia 10]
- [Huang, Lu 12]

Lemma (Dyer, Goldberg, Jerrum 09)

For complex weights, $\#\mathrm{CSP}(\mathcal{F} \cup \{[1,0],[0,1]\}) \leq_T \#\mathrm{CSP}(\mathcal{F})$.

Pl-#CSP$(\widehat{\mathscr{M}} \cup \{[1,0],[0,1]\})$ #P-hard but Pl-#CSP$(\widehat{\mathscr{M}})$ tractable

Lemma (Cai, Lu, Xia 10)

For any set of signatures $\mathcal{F}$ with real weights,

$$\text{Pl-Holant}(\widehat{\mathcal{EQ}} \cup \mathcal{F}) \text{ is \#P-hard (or in P)}$$
$$\Updownarrow$$
$$\text{Pl-Holant}(\widehat{\mathcal{EQ}} \cup \mathcal{F} \cup \{[1,0],[0,1]\}) \text{ is \#P-hard (or in P)}$$

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
- [Goldberg, Grohe Jerrum, Thurley 10]
- [Cai, Chen, Lu 10]

#CSP

- [Bulatov, Dalmau 07]
- [Dyer, Goldberg, Jerrum 09]
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09]
- [Cai, Lu, Xia 10]
- [Huang, Lu 12]

Lemma (Dyer, Goldberg, Jerrum 09)

For complex weights, $\#\mathrm{CSP}(\mathcal{F} \cup \{[1,0],[0,1]\}) \leq_T \#\mathrm{CSP}(\mathcal{F})$.

Pl-#CSP$(\widehat{\mathscr{M}} \cup \{[1,0],[0,1]\})$ #P-hard but Pl-#CSP$(\widehat{\mathscr{M}})$ tractable

Lemma (G, Williams 13)

For any set of signatures $\mathcal{F}$ with complex weights,

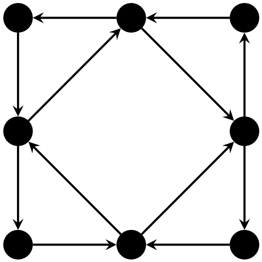
$$\text{Pl-Holant}(\widehat{\mathcal{EQ}} \cup \mathcal{F}) \text{ is \#P-hard (or in P)}$$
$$\Updownarrow$$
$$\text{Pl-Holant}(\widehat{\mathcal{EQ}} \cup \mathcal{F} \cup \{[1,0],[0,1]\}) \text{ is \#P-hard (or in P)}$$

Definition

At each vertex in an Eulerian orientation of a graph,

in-degree equals out-degree.

Example



#Pl-4Reg-EO: Theorem and Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P*-hard.*

#Pl-4Reg-EO: Theorem and Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P*-hard.*

Strengthens a theorem from [Huang, Lu 12] to the planar setting.

#Pl-4Reg-EO: Theorem and Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P*-hard.*

Strengthens a theorem from [Huang, Lu 12] to the planar setting.

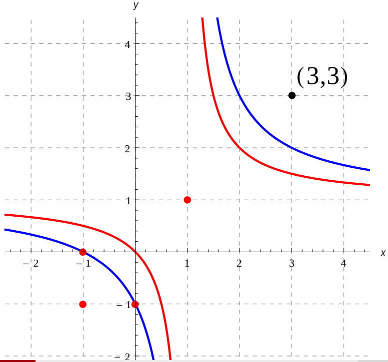
Proof.

Reduction from the evaluation of the Tutte polynomial at the point $(3, 3)$ for planar graphs:

$$\begin{aligned} \text{Pl-Tutte}(3,3) \leq_T & \quad \vdots \\ \leq_T & \ \#\text{Pl-4Reg-EO} \end{aligned}$$

Theorem (Vertigan 05)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar graphs is #P-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1, 1), (-1, -1), (j, j^2), (j^2, j)\}$, where $j = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.

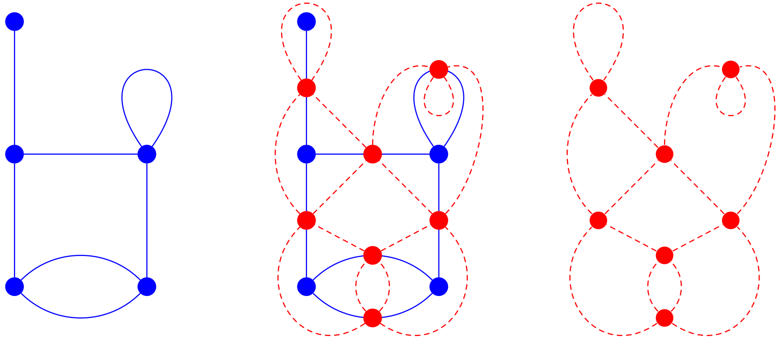


#Pl-4Reg-EO: Medial Graph

Definition

For a connected plane graph G, its medial graph H has a vertex for each edge of G and two vertices in H are joined by an edge for each face of G in which their corresponding edges occur consecutively.

Example



#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathcal{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathcal{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Theorem (Las Vergnas 88)

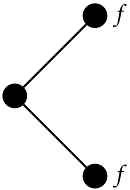
Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Pl-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

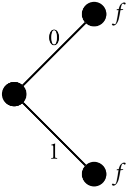
Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Pl-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

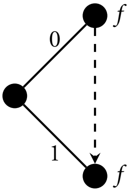
Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Pl-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

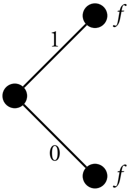
Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Pl-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



Theorem (Las Vergnas 88)

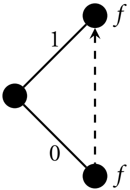
Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Pl-Holant $([0,1,0] \mid f)$

$(\neq_2) = [0,1,0]$



Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out" in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} f^{0000} & f^{0010} & f^{0001} & f^{0011} \\ f^{0100} & f^{0110} & f^{0101} & f^{0111} \\ f^{1000} & f^{1010} & f^{1001} & f^{1011} \\ f^{1100} & f^{1110} & f^{1101} & f^{1111} \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} f^{0000} & f^{0010} & f^{0001} & f^{0011} \\ f^{0100} & f^{0110} & f^{0101} & f^{0111} \\ f^{1000} & f^{1010} & f^{1001} & f^{1011} \\ f^{1100} & f^{1110} & f^{1101} & f^{1111} \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out" in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & f^{0011} \\ 0 & f^{0110} & f^{0101} & 0 \\ 0 & f^{1010} & f^{1001} & 0 \\ f^{1100} & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & f^{0011} \\ 0 & f^{0110} & f^{0101} & 0 \\ 0 & f^{1010} & f^{1001} & 0 \\ f^{1100} & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented "in, out, in, out" in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & f^{0011} \\ 0 & f^{0110} & 2 & 0 \\ 0 & 2 & f^{1001} & 0 \\ f^{1100} & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & f^{0011} \\ 0 & f^{0110} & 2 & 0 \\ 0 & 2 & f^{1001} & 0 \\ f^{1100} & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: The Connection

Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathcal{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{Pl-Tutte}_G(3,3) = \sum_{O \in \mathcal{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented ``in, out, in, out'' in cyclic order.

Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$\text{Pl-Tutte}(3,3) \equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\leq_T \quad \vdots$$

$$\leq_T \#\text{Pl-4Reg-EO}$$

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$\text{Pl-Tutte}(3,3) \equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0&0&0&1\\ 0&1&2&0\\ 0&2&1&0\\ 1&0&0&0 \end{bmatrix}\right)$$

$$\leq_T \quad \vdots$$

$$\leq_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&0 \end{bmatrix}\right)$$

$$\equiv_T \#\text{Pl-4Reg-EO}$$

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P*-hard.*

Proof.

$$\begin{aligned} \text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\ &\leq_T \qquad \vdots \\ &\leq_T \text{Pl-Holant}([0,1,0] \mid [0,0,1,0,0]) \\ &\equiv_T \#\text{Pl-4Reg-EO} \end{aligned}$$

#Pl-4Reg-EO: Holographic Transformations

To remove bipartiteness, do holographic transformation by $Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$:

#Pl-4Reg-EO: Holographic Transformations

To remove bipartiteness, do holographic transformation by $Z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$:

$$\text{Pl-Holant}\left([0,1,0] \mid f\right) \equiv_T \text{Pl-Holant}(f'),$$

where

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

#Pl-4Reg-EO: Holographic Transformations

To remove bipartiteness, do holographic transformation by $Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$:

$$\text{Pl-Holant}\left([0,1,0] \mid f\right) \equiv_T \text{Pl-Holant}(f'),$$

where

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Similarly,

$$\text{Pl-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right) \equiv_T \text{Pl-Holant}([3,0,1,0,3]).$$

#Pl-4Reg-EO: Proof Overview

Theorem

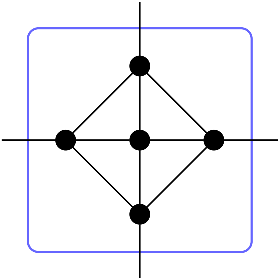
Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$
\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0&0&0&1\\ 0&1&2&0\\ 0&2&1&0\\ 1&0&0&0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 1&0&0&2 \end{bmatrix}\right) \\
&\leq_T \qquad \vdots \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right) \\
&\equiv_T \text{\#Pl-4Reg-EO}
\end{aligned}
$$

#Pl-4Reg-EO: Planar Tetrahedron Gadget

Assign $[3, 0, 1, 0, 3]$ to every vertex of this gadget...



...to get a signature $16g'$ with

$$M_{g'} = \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$
\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}\left(\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}
$$

#Pl-4Reg-EO: Proof Overview

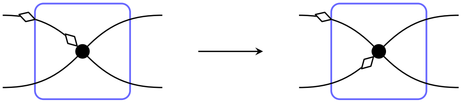
Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

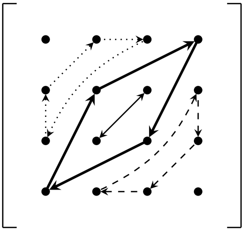
Proof.

$$
\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}\left(\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}([0,1,0] \mid [0,0,1,0,0]) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}
$$

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{g'} = \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



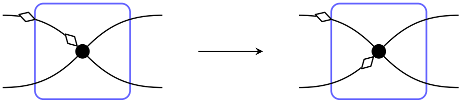
(a) A counterclockwise rotation.



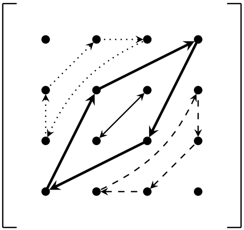
(b) Movement of signature matrix entries under a counterclockwise rotation.

#Pl-4Reg-EO: Rotationally Symmetric

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{g'} = \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



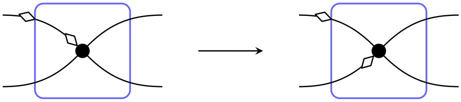
(a) A counterclockwise rotation.



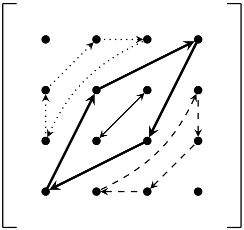
(b) Movement of signature matrix entries under a counterclockwise rotation.

#Pl-4Reg-EO: Rotationally Symmetric

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{g'} = \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



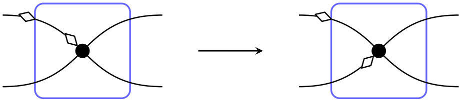
(a) A counterclockwise rotation.



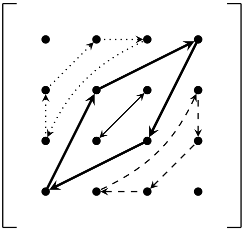
(b) Movement of signature matrix entries under a counterclockwise rotation.

#Pl-4Reg-EO: Rotationally Symmetric

$$M_{f'} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{g'} = \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



(a) A counterclockwise rotation.



(b) Movement of signature matrix entries under a counterclockwise rotation.

#Pl-4Reg-EO: Diagonalization

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

#Pl-4Reg-EO: Diagonalization

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Then

$$M_{f'} = T\Lambda_{f'}T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1}$$

and

$$M_{g'} = T\Lambda_{g'}T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

#Pl-4Reg-EO: Diagonalization

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Then

$$M_{f'} = T \Lambda_{f'} T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1}$$

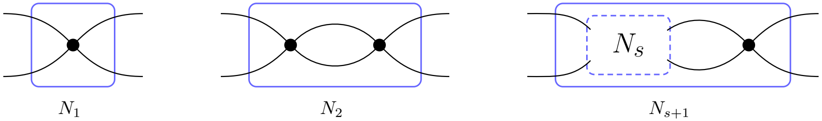
and

$$M_{g'} = T \Lambda_{g'} T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

Follows from being both rotationally symmetric and complement invariant.

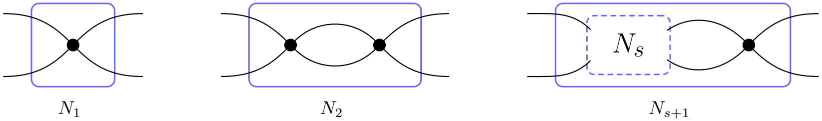
#Pl-4Reg-EO: Interpolation

Suppose that f' appears n times in Ω of Pl-Holant(f').
Construct instances Ω_s of Holant(g') indexed by $s \geq 1$.
Obtain Ω_s from Ω by replacing each f' with N_s (g' assigned to all vertices).



#Pl-4Reg-EO: Interpolation

Suppose that f' appears n times in Ω of Pl-Holant(f').
Construct instances Ω_s of Holant(g') indexed by $s \geq 1$.
Obtain Ω_s from Ω by replacing each f' with N_s (g' assigned to all vertices).



To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
   we first replace $M_{f'}$ with $T \Lambda_{f'} T^{-1}$. (Holant unchanged)

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
   we first replace $M_{f'}$ with $T\Lambda_{f'} T^{-1}$. (Holant unchanged)
2. Then we replace $T\Lambda_{f'} T^{-1}$ with $T(\Lambda_{g'})^s T^{-1}$.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
   we first replace $M_{f'}$ with $T\Lambda_{f'}T^{-1}$. (Holant unchanged)
2. Then we replace $T\Lambda_{f'}T^{-1}$ with $T(\Lambda_{g'})^sT^{-1}$.

We only need to consider the assignments to $\Lambda_{f'}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
we first replace $M_{f'}$ with $T\Lambda_{f'}T^{-1}$. (Holant unchanged)
2. Then we replace $T\Lambda_{f'}T^{-1}$ with $T(\Lambda_{g'})^s T^{-1}$.

We only need to consider the assignments to $\Lambda_{f'}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
   we first replace $M_{f'}$ with $T\Lambda_{f'}T^{-1}$. (Holant unchanged)
2. Then we replace $T\Lambda_{f'}T^{-1}$ with $T(\Lambda_{g'})^s T^{-1}$.

We only need to consider the assignments to $\Lambda_{f'}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

To obtain Ω_s from Ω,
we effectively replace $M_{f'}$ with $M_{N_s} = (M_{g'})^s$.

1. To obtain Ω_s from Ω,
   we first replace $M_{f'}$ with $T\Lambda_{f'}T^{-1}$. (Holant unchanged)
2. Then we replace $T\Lambda_{f'}T^{-1}$ with $T(\Lambda_{g'})^sT^{-1}$.

We only need to consider the assignments to $\Lambda_{f'}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω.

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

Then

$$\text{Pl-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^{\ell} c_{jk\ell}$$

#Pl-4Reg-EO: Interpolation

$$\Lambda_{f'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{g'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

Then

$$\text{Pl-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^{\ell} c_{jk\ell}$$

and

$$\text{Pl-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (6^{k} 13^{\ell})^{s} c_{jk\ell}$$

is a full rank Vandermonde system (row index s, column index $c_{jk\ell}$).

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$
\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}\left(\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}\right) \\
&\leq_T \text{Pl-Holant}([3,0,1,0,3]) \\
&\equiv_T \text{Pl-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}
$$

□

#Pl-4Reg-EO: Proof Overview

Theorem

Counting Eulerian Orientations for planar 4-regular graphs is #P-hard.

Proof.

$$\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0&0&0&1\\ 0&1&2&0\\ 0&2&1&0\\ 1&0&0&0 \end{bmatrix}\right)\\
&\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 2&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 1&0&0&2 \end{bmatrix}\right)\\
&\leq_T \text{Pl-Holant}\left(\begin{bmatrix} 19&0&0&7\\ 0&7&5&0\\ 0&5&7&0\\ 7&0&0&19 \end{bmatrix}\right)\\
&\leq_T \text{Pl-Holant}([3,0,1,0,3])\\
&\equiv_T \text{Pl-Holant}\left([0,1,0] \mid [0,0,1,0,0]\right)\\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}$$

□

Major proof techniques:

1. Holographic transformation
2. Gadget construction
3. Interpolation

Thank You