A Holant Dichotomy:

Is the FKT Algorithm Universal?

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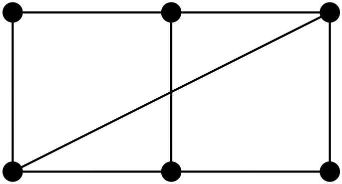
Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$



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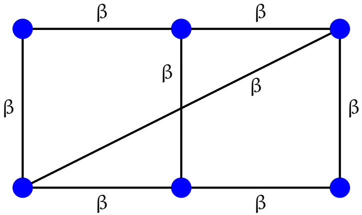

Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^8$$

$$\Pr(\sigma) \sim w(\sigma)$$

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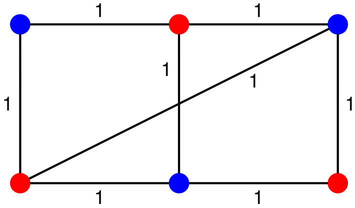

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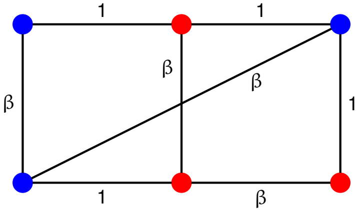

Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^4$$

$$\Pr(\sigma) \sim w(\sigma)$$

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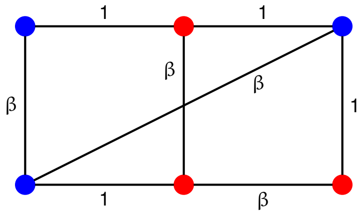

Partition function (normalizing factor):

$$Z_G(\beta) = \sum_{\sigma: V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ.

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- Reduction to #PM (counting perfect matchings) in planar graphs.
  - #PM is #**P**-hard [Valiant 79] in general graphs as well.
- #PM can be computed via Pfaffian orientations of planar graphs.

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We need to answer this question in some framework.

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- **Name** #CSP$(\mathcal{F})$

  Instance A bipartite graph $G = (V, C, E)$ and a mapping $\pi : C \to \mathcal{F}$

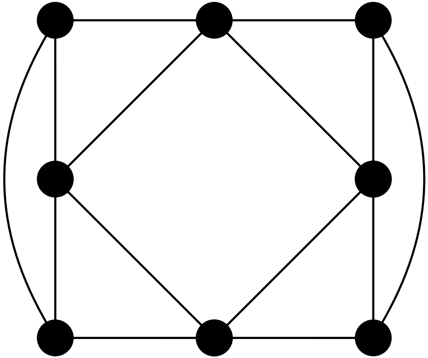
  Output The quantity:

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

  where $N(c)$ are the neighbors of c and $f_c = \pi(c) \in \mathcal{F}$.

Counting Perfect Matchings

Perfect Matchings



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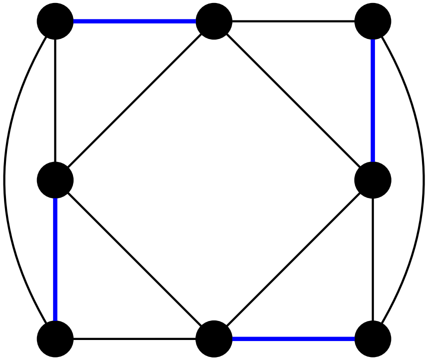

Holant Problems

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- **Name** Holant($\mathcal{F}$)

  Instance A graph $G = (V, E)$ and a mapping $\pi : V \to \mathcal{F}$

  Output The quantity:

$$\sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v\left(\sigma \mid_{E(v)}\right),$$

where $E(v)$ are the incident edges of v and $f_v = \pi(v) \in \mathcal{F}$.

- More general than #CSP:

$$\#\text{CSP}(\mathcal{F}) \equiv_T \text{Holant}(\mathcal{EQ} \cup \mathcal{F}),$$

  where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.
- Equivalent formulation: Tensor network contraction . . .
- Pl-Holant($\mathcal{F}$) denotes the version where instances are all planar.

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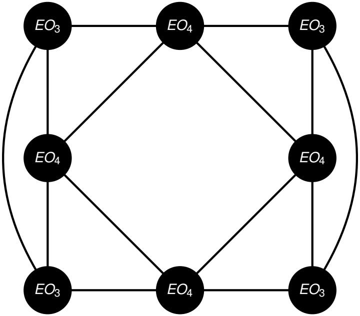
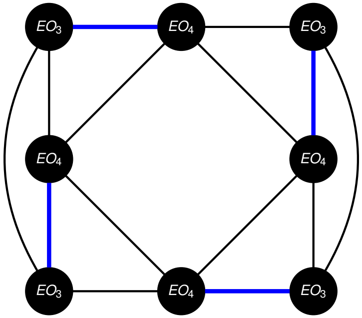


#PM as a Holant

- Put functions ExactOne (EO) on nodes (edges are variables).
- #PM is then the partition function:

$$\#\text{PM} = \sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} \text{EO}_d(\sigma |_{E(v)}).$$



Complexity Classifications

Counting problems with local constraints are usually classified into:

1. **P**-time solvable over general graphs;
2. #**P**-hard over general graphs but **P**-time solvable over planar graphs;
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Category (2) is always captured by holographic algorithms with matchgates.

Examples include:

- Tutte polynomials [Vertigan 91], [Vertigan 05].
- Spin systems [Kowalczyk 10], [Cai, Kowalczyk, Williams 12].
- #CSP [Cai, Lu, Xia 10], [G. and Williams 13].

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Category (1) is characterized in [Cai, G., Williams 13].

Category (3) is not captured by holographic algorithms with matchgates!

New Planar Tractable Case

Counting Orientations, where two types of nodes are allowed:

1. Exactly one edge coming in;
2. All edges coming in or going out (either a sink or a source).

Moreover, we require that the gcd of the degrees of type 2 nodes is at least 5.

Then the problem is tractable.

#PM in Planar Hypergraphs

As a special case of our result, consider the following problem.

Name #Planar-Hyper-PM(S)

Instance A hypergraph H whose incidence graph is planar, and hyperedge sizes are prescribed by S.

Output The number of perfect matchings in H.

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Let $t = \gcd(S)$.

- If $t \geqslant 5$ or $S \subseteq \{1, 2\}$,

  then #Planar-Hyper-PM(S) is computable in polynomial time.

- Otherwise $t \leqslant 4$, $S \not\subseteq \{1, 2\}$, and #Planar-Hyper-PM(S) is #**P**-hard.

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- The planar constraint guarantees the existence of certain structures that can be simplified.
- Some steps of the process may provide orientations inconsistent with the original instance, but we can keep track of enough information to go back and check.
- Tractable mainly due to degree rigidity.

Examplary Planar Structures

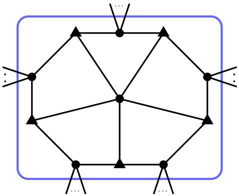
Lemma

Let $G = (L \cup R, E)$ be a planar bipartite graph with parts L and R.

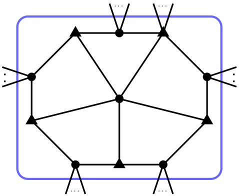
Every vertex in L has degree at least 5;

every vertex in R has degree at least 3.

If G is simple, then there exists one of the two wheel structures in G.



(a) Type 1



(b) Type 2

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- Assign a *score* s_v to each vertex $v \in V$ so that

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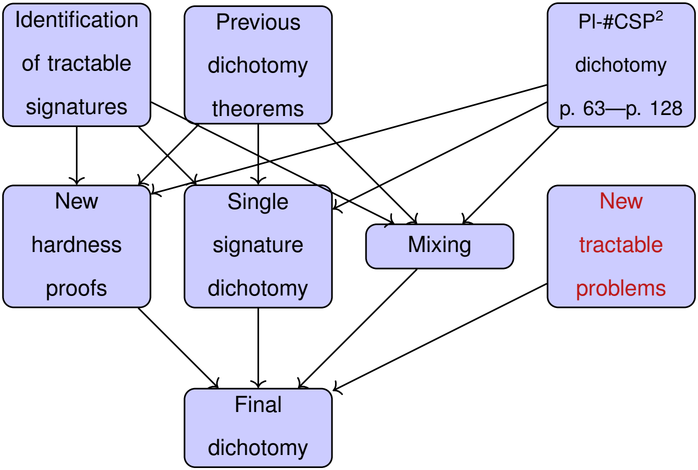
- If no wheel structure exists, then there exists a 1-1 mapping between positive vertices and negative vertices, and negative scores are larger. Hence the total score has to be negative. Contradiction.

Proof Roadmap of the Main Theorem



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- The natural generalization for $d \geqslant 5$ does not hold, and in the end we proved the $d = 2$ case (where the natural generalization does hold).
- However lots of progress was made due to this belief.

Concluding Remarks

- A sharp algebraic separation exists between tractable and #**P**-hard problems.

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- There exists planar tractable cases that are not captured by holographic algorithms with matchages (or FKT).

Thank You!