Uniqueness, Spatial Mixing, and Approximation for Ferromagnetic 2-Spin Systems

Heng Guo[1] and Pinyan Lu[2]

[1]Queen Mary, University of London

[2]Shanghai University of Finance and Economics

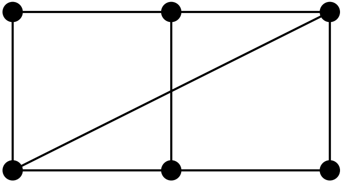
Paris, France

Sep 08 2016

Ising Model

Edge interaction

	0	1
0	β	1
1	1	β



Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^{mono(\sigma)}$$

$$\pi(\sigma) \sim w(\sigma)$$

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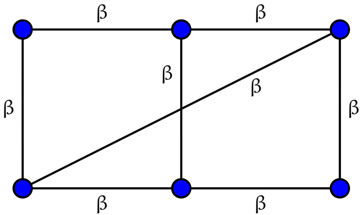

Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^8$$

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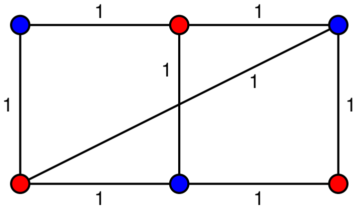

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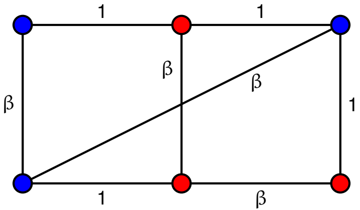

Partition function (normalizing factor):

$$Z_G(\beta) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{mono(\sigma)}$, $mono(\sigma)$ is the number of monochromatic edges under σ.

2-State Spin System

Edge:

	0	1
0	β	1
1	1	β

Vertex:

0	1
1	1

More generally, three parameters β, γ, and λ.

$$w(\sigma) = \beta^{m_0(\sigma)}\gamma^{m_1(\sigma)}\lambda^{n_0(\sigma)}$$

$m_0(\sigma)$: # of $(0,0)$ edges;

$m_1(\sigma)$: # of $(1,1)$ edges;

$n_0(\sigma)$: # of 0 vertices.

$$Z_G(\beta,\gamma,\lambda) = \sum_{\sigma:V\to\{0,1\}} w(\sigma)$$

2-State Spin System

Edge: $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ Vertex: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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Examples

- Ising model: $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (no external field)

$$Z_G(\beta) = \sum_{\sigma:V\to\{0,1\}} \beta^{mono(\sigma)}$$

- Hardcore gas model: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ (Weighted independent set)

$$Z_G(\beta) = \sum_{\text{Independent set } I} \lambda^{|I|}$$

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Approximate Counting

- Exact evaluating Z is **#P**-hard unless $\beta\gamma = 1$ or $\beta = \gamma = 0$ or $\lambda = 0$.
- Approximate the partition function Z.
  - Fully Polynomial-time Randomized Approximation Scheme (FPRAS) and FPTAS:
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Ferromagnetic and Anti-ferromagnetic

Edge Interaction

$$\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

- If $\beta\gamma = 1$, then the 2-spin system is trivial.

- Ferromagnetic Ising: $\beta\gamma > 1$.

  Neighbours tend to have the same spin.

- Anti-ferromagnetic Ising: $\beta\gamma < 1$.

  Neighbours tend to have different spins.

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Ferromagnetic 2-Spin Systems

Previous Work

- FPRAS exists for ferromagnetic Ising models with consistent fields:
  $\beta = \gamma > 1$ and $\lambda_v \geqslant 1$ (or $\leqslant 1$) for all $v \in V$
  [Jerrum, Sinclair 93].

- Extended to $\lambda_v \leqslant \frac{\gamma}{\beta}$ (if $\beta \leqslant \gamma$ and $\beta\gamma > 1$)
  [Goldberg, Jerrum, Paterson 03], [Liu, Lu, Zhang 14].

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Main Theorem

Theorem

If $\beta \leqslant 1 \leqslant \gamma$, $\beta\gamma > 1$, *and* $\lambda_v \leqslant \lambda_c = \left(\frac{\gamma}{\beta}\right)^{\Delta_c/2}$ *where* $\Delta_c = \frac{2\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1}$, *then FPTAS exists.*

- If we allow $\lambda_v > \lambda_c^{int} = \left(\frac{\gamma}{\beta}\right)^{(\lfloor\Delta_c\rfloor+1)/2}$,

  then Z is #BIS-hard to approximate [Liu, Lu, Zhang 14].
- #BIS is the complexity upper bound for all ferro 2-spin systems.

Ferro 2-Spin

Ferro 2-spin systems: Edge: $\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$ Vertex: $\begin{bmatrix} \lambda_v \\ 1 \end{bmatrix}$

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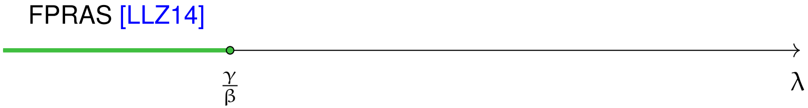
For general graph G, assuming $\beta \leqslant \gamma$:



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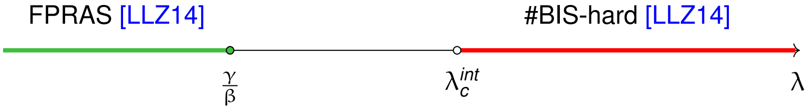

$$\lambda_c^{int} = \left(\frac{\gamma}{\beta}\right)^{(\lfloor \Delta_c \rfloor + 1)/2}, \text{ where } \Delta_c = \frac{2\sqrt{\beta\gamma}}{\sqrt{\beta\gamma} - 1}.$$

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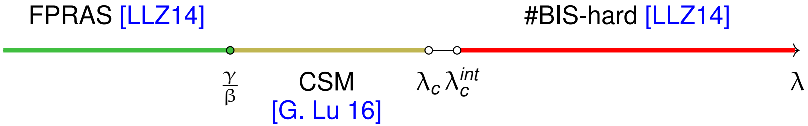

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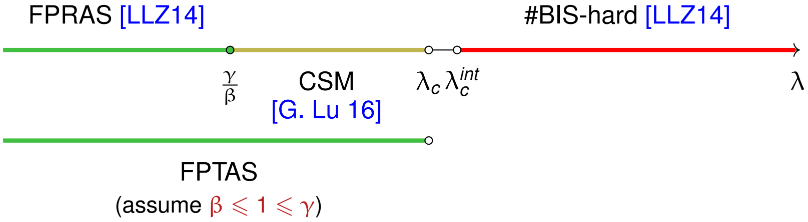

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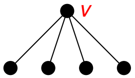
Weitz's Correlation Decay Algorithm

Goal: calculate marginal probabilities using tree recursions.

Replace a vertex of degree *d* with *d* copies.

$$R_v = \frac{\Pr(v = 0)}{\Pr(v = 1)}$$



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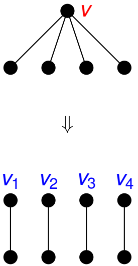

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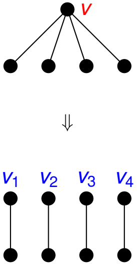
$$= \frac{\Pr(0000)}{\Pr(0001)} \cdot \frac{\Pr(0001)}{\Pr(0011)} \cdot \frac{\Pr(0011)}{\Pr(0111)} \cdot \frac{\Pr(0111)}{\Pr(1111)}$$



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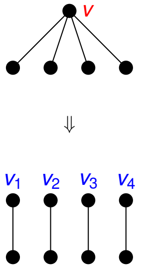
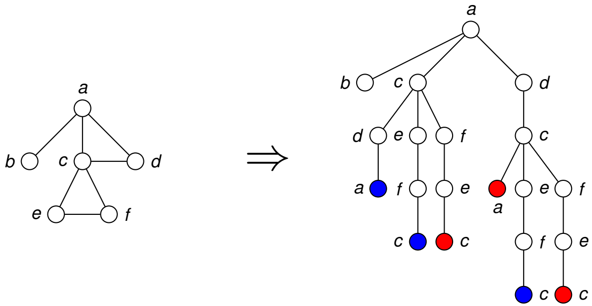


Each term $\frac{\Pr(0011)}{\Pr(0111)}$ can be viewed as the marginal ratio of v_i conditioned on a certain configuration of other v_j's.

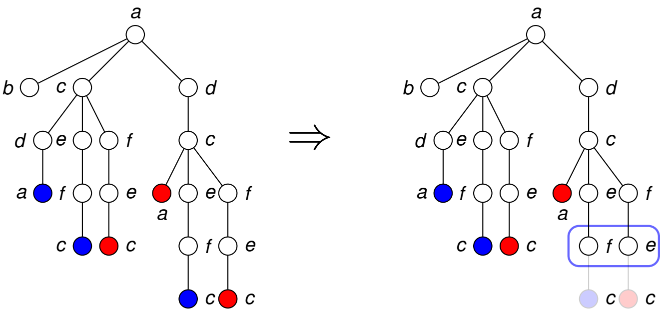
Self-Avoiding Walk (SAW) Tree

- SAW tree is essentially the tree of self-avoiding walks originating at v except that the vertices closing a cycle are also included in the tree.
  - Cycle-closing vertices are fixed according to the rule in the last slide.
- Do the tree recursion to calculate p_v.

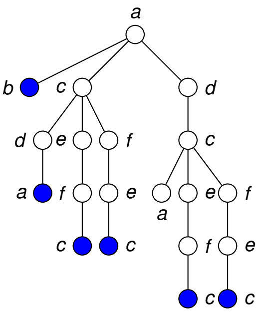


Correlation Decay

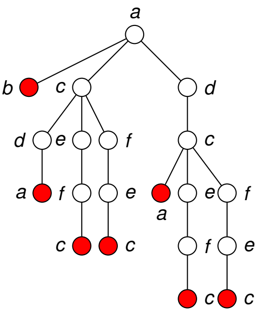
- SAW tree has exponential size in general.
  - Truncate the recursion within logarithmic depth.
  - How much error does the truncation incur?



Weak Spatial Mixing:

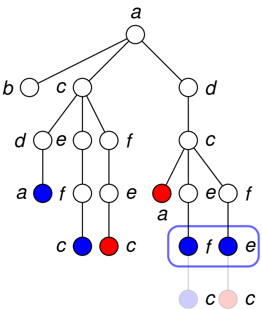


V.S.



Spatial Mixing in Trees

Strong Spatial Mixing:



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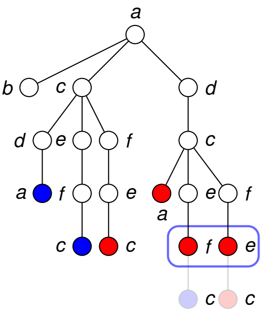

Conditional Spatial Mixing

If $\lambda_v < \lambda_c$ for all v, conditional spatial mixing holds in arbitrary trees:

Instead of worst case configurations in SSM, we only allow partial configurations that are dominated by the product measure of isolated vertices ($p_v \leqslant \frac{\lambda}{1+\lambda}$).

(All vertices are leaning towards the good spin.)

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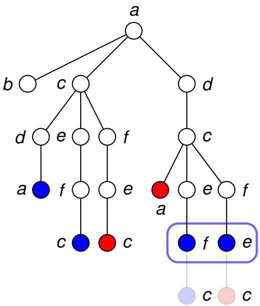

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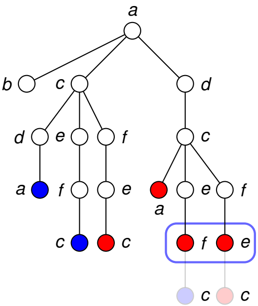

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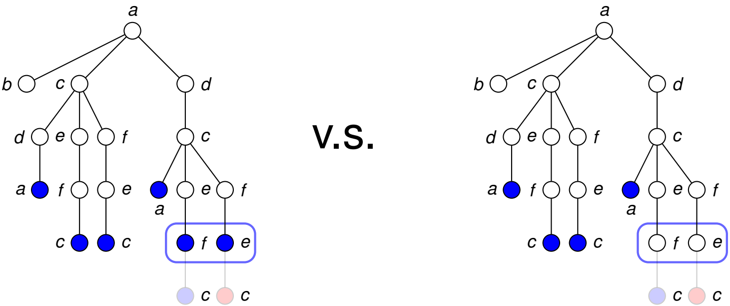
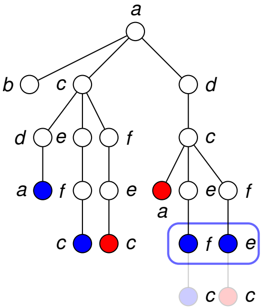


Pruning

If $\beta \leqslant 1 < \gamma$, in the SAW tree, we may first remove "bad" pinnings,

the effective field is smaller (better).



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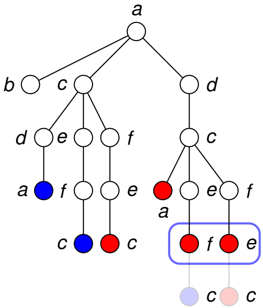

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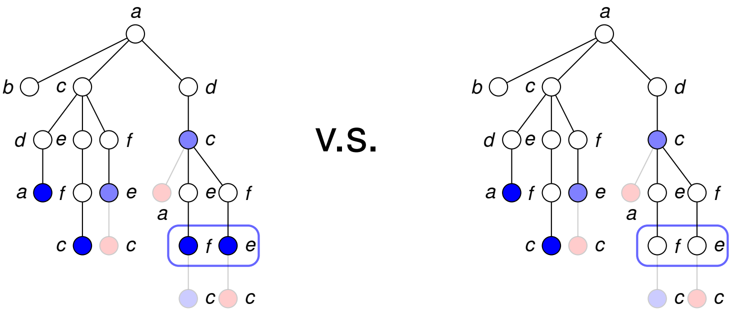
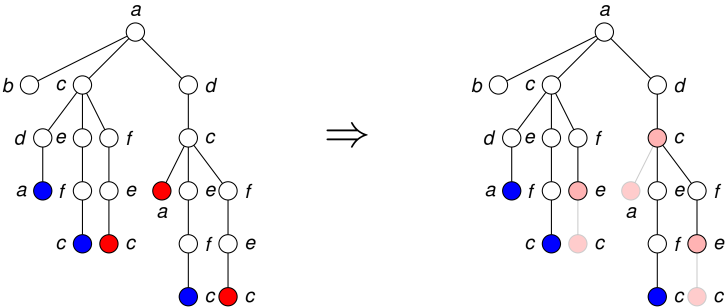


CSM $\Rightarrow$ SSM

If $\beta > 1$, then pruning fails.

In fact, there is no λ such that SSM holds for general trees.



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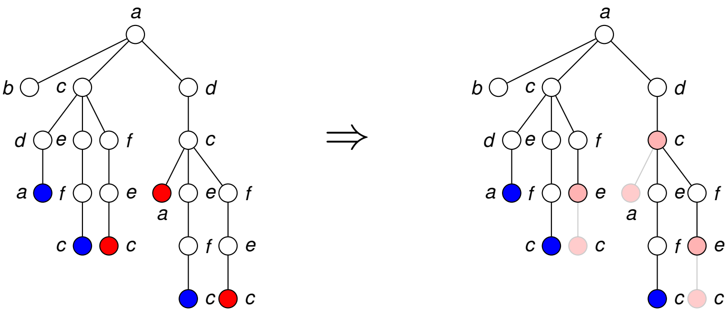

However, if $\lambda_v \leqslant \lambda_c$, then $p_v \leqslant \frac{\lambda}{1+\lambda}$ for any graph G.

FPTAS without SSM?

The Exact Threshold?

Our result is tight up to an integrality gap.

However, neither λ_c nor λ_c^{int} is the right bound.

- There exists a small interval beyond λ_c where FPTAS still exists.
  - Degrees have to be integers.
- There is a $\lambda < \lambda_c^{int}$ such that SSM fails (in an irregular tree).

$$\text{WSM (in } \mathbb{T}_\Delta) \not\Rightarrow \text{SSM}$$

(even if $\beta \leqslant 1 < \gamma$)

This is in contrast to the anti-ferro case.

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Open Questions

Ferro 2-spin systems:

- FPTAS for $1 < \beta \leqslant \gamma$, $\lambda_v < \lambda_c$?
  - Conditional spatial mixing for graphs instead of trees.

    (This implies FPTAS for, say, planar graphs.)
- Avoid the gadget gap in the hardness proof.

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