University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Homework 3

Problem 1: Project the volume of a d-dimensional ball of radius $\sqrt{d}$ onto a line through the centre. For large d, give an intuitive argument that the projected volume should behave like a Gaussian.

Problem 2: Consider a nonorthogonal basis $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_d}$. The $\mathbf{e_i}$ are a set of linearly independent unit vectors that span the space.

1. Prove that the representation of any vector in this basis is unique;
2. Calculate the squared length of $\mathbf{z} = \frac{\sqrt{2}}{2}\mathbf{e_1} + \mathbf{e_2}$ where $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
3. If $\mathbf{y} = \sum_{i=1}^{d} a_i \mathbf{e_i}$ and $\mathbf{z} = \sum_{i=1}^{d} b_i \mathbf{e_i}$, with $0 < a_i < b_i$ for all $1 \le i \le d$, is it necessarily true that the length of $\mathbf{z}$ is grater than the length of $\mathbf{y}$? If yes give a proof of the statement, if no find a counterexample;
4. Consider the basis $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
   (a) What is the representation of the vector $\mathbf{v} = (0, 1)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$? I.e. find scalars a, b such that $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2}$.
   (b) What is the representation of the vector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$?
   (c) What is the representation of the vector $(1, 2)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$?

Problem 3: Compute the right-singular vectors v_i, the left-singular vectors u_i, the singular values σ_i and hence find the *Singular value decomposition* of

1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}$;
2. $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{pmatrix}$.