University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Homework 4

Problem 1: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \\ -1 & -2 \end{pmatrix}.$$

1. Run the *power method* starting from $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $k = 3$ steps. What does this give as estimates for v_1 and σ_1?
2. What are the actual values of v_i's, σ_i's and u_i's? You might find it helpful to first compute the eigenvalues and eigenvectors of $B = A^\intercal A$.
3. Suppose matrix A is a database of restaurant ratings: each row corresponds to a person, each column to a restaurant, and the entries A_{ij} represent how much person i likes restaurant j. What might v_1 represent? What about u_i? What about the gap $\sigma_1 - \sigma_2$?

Problem 2: Let $v \in \mathbb{R}^n$ such that $\|v\| = 1$. Sample uniformly $x \in \{-1, 1\}^n$, and define $S = \langle x, v \rangle$. Prove that

$$\mathbf{E}\left[S^4\right] = 3\sum_{i=1}^{n} v_i^2 - 2\sum_{i=1}^{n} v_i^4 \le 3.$$

That is, prove the inequality from the Proof of Lemma 2 in Lecture 6.

Problem 3: Let $A \in \mathbb{R}^{n\times n}$ be a symmetric and PSD matrix. Show that the power method can be applied to approximately compute the smallest eigenvalue of A.

Problem 4: Let u be a fixed vector. Show that maximising $x^\intercal u u^\intercal (1 - x)$ subject to $x_i \in \{0, 1\}$ is equivalent to partitioning the coordinates of u into two subsets where the sum of the elements in both subsets are as equal as possible.

Problem 5 *(Optional)*: Let $x_1, x_2, \ldots, x_n$ be n points in a d-dimensional space and let X be an $n \times d$ matrix whose rows are the n points. Suppose we know only the matrix D of pairwise distances between points and not the coordinates of the points themselves. The set of points $x_1, x_2, \ldots, x_n$ giving rise to the matrix D is not unique since any translation, rotation or reflection of the coordinate system preserves the distances. Fix the origin of the coordinate system so that the centroid of the set of points is at the origin. That is, $\sum_{i=1}^{n} x_i = 0$.

1. Show that the elements of $XX^\intercal$ are given by

$$x_i^\intercal x_j = -\frac{1}{2}\left(d_{ij}^2 - \frac{1}{n}\sum_{k=1}^{n} d_{ik}^2 - \frac{1}{n}\sum_{k=1}^{n} d_{kj}^2 + \frac{1}{n^2}\sum_{k=1}^{n}\sum_{\ell=1}^{n} d_{k\ell}^2\right).$$

2. Describe an algorithm for determining the matrix X whose rows are the x_i.