University of Edinburgh
INFR11156: Algorithmic Foundations of Data Science (2019)
Homework 7

Problem 1: Prove that G has exactly k connected components if and only if $\lambda_k = 0$ and $\lambda_{k+1} > 0$.

Problem 2: Prove that, for any connected graph G with diameter α, it holds that

$$\lambda_2 \geq \frac{1}{\alpha \cdot \mathrm{vol}(G)}.$$

Problem 3: Prove that for any graph G, $\lambda_2 \leq n/(n-1)$ and $\lambda_2 = n/(n-1)$ if and only if G is the complete graph.

Problem 4: [1] Let G be an undirected and connected d-regular graph, and let $\mathbf{A}$ be its adjacency matrix with the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$.

1. Prove that $\lambda_n = -d$ if and only if G is bipartite.
2. Assume that $G = (V, W, E)$ is bipartite and $\max_{2 \leq i \leq n-1} |\lambda_i| = \mu$. For any $S \subseteq V$ and $T \subseteq W$, let $e(S, T)$ be the number of edges between S and T. Prove that

$$e(S,T) \leq \frac{2d|S||T|}{|S| + |T|} + \mu n.$$

Problem 5 (challenging): Let $G = (V, E)$ be an undirected graph with m edges. Prove that, for any fixed k, the number of cycles of length k in G is at most $O(m^{k/2})$.

[1] This problem appeared in the last year's AFDS final exam. You should be able to answer questions of a similar level of difficulty in your final exam.