University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Homework 8

Problem 1: Prove the following Courant-Fischer Min-Max Characterisation of Eigenvalues. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding eigenvectors $f_1, \ldots, f_n$. Then, it holds for any $1 \leq i \leq n$ that

$$\lambda_i = \min_{S:\dim S = i} \max_{x\in S, x\neq 0} \frac{x^\intercal A x}{x^\intercal x} = \max_{S:\dim S = n-i+1} \min_{x\in S, x\neq 0} \frac{x^\intercal A x}{x^\intercal x},$$

where S is a subspace of $\mathbb{R}^n$.

Problem 2 (challenging): We know that every rule for clustering must display some strange behaviour. In this problem, you will prove this for partitioning a weighted graph $G = (V, E, w)$ by minimising the conductance h_G. In particular, you will consider dividing a graph into two pieces by finding the set $S \subseteq V$ with $\mathrm{vol}(S) \leq \mathrm{vol}(V)/2$ minimising

$$h_G(S) \triangleq \frac{w(S, V\setminus S)}{\mathrm{vol}(S)}.$$

You need to show that it is possible to split a cluster by adding an edge to the cluster or by increasing the weight of an edge inside the cluster. That is, construct a graph G so that if S is the set minimising h_G, there is an edge you can add between the vertices of S, or an edge between the vertices of S whose weight you can increase, so that after you do this the set S' minimising h_G is a proper subset of S. This goal will consists of the following two tasks:

1. Describe your graph G, the set S minimising h_G, and prove your claim.
2. Describe the edge you add or whose weight you increase to produce a new graph G'; describe the set S' minimising $h_{G'}$ in the modified graph and prove your claim.