University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Homework 9

Problem 1: Let $G = (V, E)$ be an arbitrary undirected graph with n vertices.

1. Assume that H_1 and H_2 are $(1+\varepsilon)$-spectral sparsifiers of G with corresponding Laplacian matrices L_{H_1} and L_{H_2}. Prove or disprove by counterexample the following statement: "The graph defined by the Laplacian matrix $(1/2) \cdot (L_{H_1} + L_{H_2})$ is a $(1+\varepsilon)$-spectral sparsifier of G as well."

2. Assume that $G = (V, E)$ is an unweighted and complete graph. Prove or disprove the following statement: "The graph consisting of $O(n \log n/\varepsilon^2)$ edges, where every edge is sampled uniformly at random from G, is a $(1+\varepsilon)$-spectral sparsifier of G. "

3. Assume that $G = (V, E)$ is an arbitrary unweighted graph. Prove or disprove the following statement: "The graph consisting of $O(n \log n/\varepsilon^2)$ edges, where every edge is sampled uniformly at random from G, is a $(1+\varepsilon)$-spectral sparsifier of G. "

4. Prove or disprove by counterexample the following statement: "If H is a $(1+\varepsilon)$-spectral sparsifier of $G = (V, E)$, then for any subset $S \subseteq V$, the cut value between S and $V \setminus S$ in G and the one in H is approximately the same up to a multiplicative factor of $(1 \pm \varepsilon)$."

Problem 2: Let H be a $(1+\varepsilon)$-spectral sparsifier of graph $G = (V, E, w)$ for some constant $\varepsilon \in (0, 1/3)$. Prove that, for any set $S \subset V$, the conductance $\phi_H(S)$ of set S in H and the one $\phi_G(S)$ satisfies

$$\phi_H(S) \leq (1 + 3\varepsilon)\phi_G(S).$$