Hashing Functions

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Challenge. Universe U can be extremely large so defining an array b is infeasible.

Applications. File systems, databases, networks, cryptography, web caching.

Hashing

Hash function. $h : U \to [n]$, where $[n] := \{0, 1, \ldots, n-1\}$.

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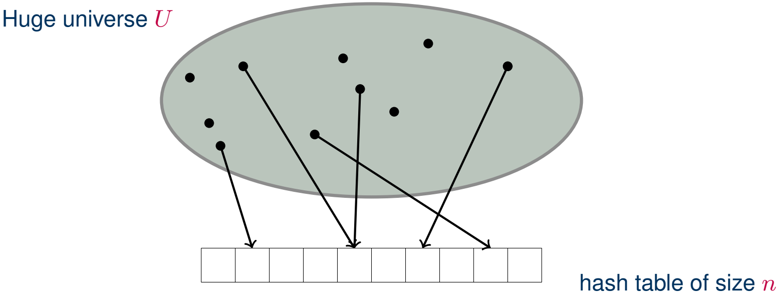
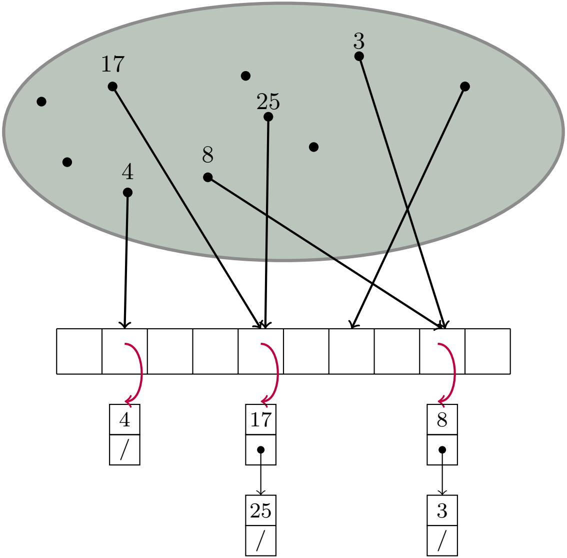
birthday paradox

Collision. When $h(u) = h(v)$ but $n \neq v$.

- collision is expected after $\Theta(\sqrt{n})$ random insertions.
- Separate chaining: $a[i]$ stores linked list of elements u with $h(u) = i$.



Hashing with chaining



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Approach. Use randomisation for the choice of h.

adversary knows the randomised algorithm you are using, but doesn't know random choices that the algorithm makes.

Universal hashing (Carter-Wegman 1980s)

A universal family of hash functions is a set of hash functions H mapping a universe U to the set $\{0, 1, \ldots, n-1\}$ such that, for any pair of elements $u \neq v$,

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Example. $U = \{a, b, c, d, e, f\}, n = 2$

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
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Universal hashing: analysis

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Q: How can we design a universal class of hash functions?

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$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \mod p.$$

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- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff $\sum_{i=1}^r a_i x_i \equiv \sum_{i=1}^r a_i y_i \mod p$, i.e.,

$$a_j \underbrace{(y_j - x_j)}_{z} \equiv \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_{m} \mod p$$

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- Since p is prime, $a_j z \equiv m \mod p$ has at most one solution among p possibilities. $\Leftarrow$ See lemma on the next slide.
- Thus $\mathbb{P}[h_a(x) = h_a(y)] \leq 1/p$.

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Let p be prime, and let $z \not\equiv 0 \mod p$. Then $az \equiv m \mod p$ has at most one solution $0 \leq a < p$.

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Proof. The proof is by contradiction.

- Suppose $0 \leq a_1 < p$ and $0 \leq a_2 < p$ are two different solutions.

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Let p be prime, and let $z \not\equiv 0 \mod p$. Then $az \equiv m \mod p$ has at most one solution $0 \leq a < p$.

Proof. The proof is by contradiction.

- Suppose $0 \leq a_1 < p$ and $0 \leq a_2 < p$ are two different solutions.
- Then $(a_1 - a_2)z \equiv 0 \mod p$; hence $(a_1 - a_2)z$ is divisible by p.

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- This implies $a_1 = a_2$.

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Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Universal hashing: summary

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$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i\right) \mod p$$

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Consequence.

- Space used $= \Theta(n)$.
- Expected number of collisions per operation is ≤ 1.

$\Rightarrow O(1)$ time per insert, delete, or lookup

Applications of hashing: finger printing

Problem. Suppose there are two documents X and Y located at two different places, and we want to know if these two documents are the same.

Applications of hashing: finger printing

Problem. Suppose there are two documents X and Y located at two different places, and we want to know if these two documents are the same.

A naive solution. Send two documents to the same place, and make a deterministic comparison.

This method has zero-error, but produces high communication cost.

Applications of hashing: finger printing (cont.)

An alternative solution. Use a universal hash function h to map each document to a k-bit string. We only need to send h, and $h(X)$(or $h(Y)$) instead.

Generate h & send $h, h(X)$ $\longrightarrow$

$\longleftarrow$ return if $h(X) = h(Y)$

Applications of hashing: finger printing (cont.)

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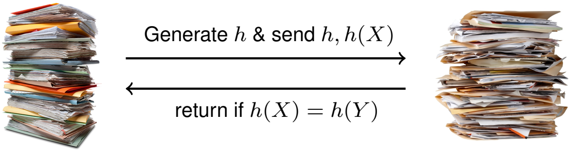

Analysis of the error probability.

$$
\begin{aligned}
\mathbb{P}_{h\in H}[\text{err}] &= \mathbb{P}_{h\in H}[h(X) \neq h(Y) | X = Y] + \mathbb{P}_{h\in H}[h(X) = h(Y) | X \neq Y] \\
&= 0 + \mathbb{P}_{h\in H}[h(X) = h(Y) | X \neq Y] \\
&\leq 1/2^k.
\end{aligned}
$$

Pairwise independent hash functions

PAIRWISE INDEPENDENCE

A family of functions $H = \{h \mid h : U \mapsto [n]\}$ is pairwise independent if, for any h chosen uniformly at random from H, the following holds:

1. $h(x)$ is uniformly distributed in $[n]$ for any $x \in U$;
2. For any $x_1 \neq x_2 \in U$, $h(x_1)$ and $h(x_2)$ are independent.

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These two conditions state that for any different $x_1 \neq x_2 \in U$, and any $y_1, y_2 \in [n]$, it holds that

$$\mathbb{P}_{h \in \mathcal{H}}\left[h(x_1) = y_1 \wedge h(x_2) = y_2\right] = \frac{1}{n^2},$$

where the probability above is over all random choices of a function from H.

Construction of pairwise independent hash functions

THEOREM

Let p be a prime number, and let $h_{a,b} = (ax+b) \bmod p$. Define

$$H = \{h_{a,b} \mid 0 \leq a, b \leq p-1\}.$$

Then H is a family of pairwise independent hash functions.

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Recall $\mathbb{Z}_p = \{0, 1, \ldots, p - 1\}$

Proof. We need to show that, for any two $x_1 \neq x_2 \in \mathbb{Z}_p$ and any $y_1, y_2 \in \mathbb{Z}_p$, it holds

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For any a, b, the conditions that $h_{a,b}(x_1) = y_1$ and $h_{a,b}(x_2) = y_2$ yield two equations

$$ax_1 + b = y_1 \mod p,$$

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Such system has a unique solution of a and b, out of p^2 possible pairs of (a, b). Hence, the equation above holds.

Generalisation: k-wise independence

The set $H = \{h : U \to [n]\}$ is call a set of k-wise independent family of hash functions if for any distinct $x_1, \ldots, x_k \in U$, and any $y_1, \ldots, y_k \in [n]$,

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Construction of k-wise hash functions

Let p be a prime, and $k \geq 2$ be an integer. Assume that a seed $s = (a_0, \ldots, a_{k-1})$ is chosen uniformly at random from $\mathbb{Z}_p^k$. Then, the set of functions $H = \left\{h_s | s \in \mathbb{Z}_p^k\right\}$, where

$$h_s(x) = \sum_{i=0}^{k-1} a_i x^i \mod p$$

is k-wise independent.