Data streaming algorithms (1)

He Sun

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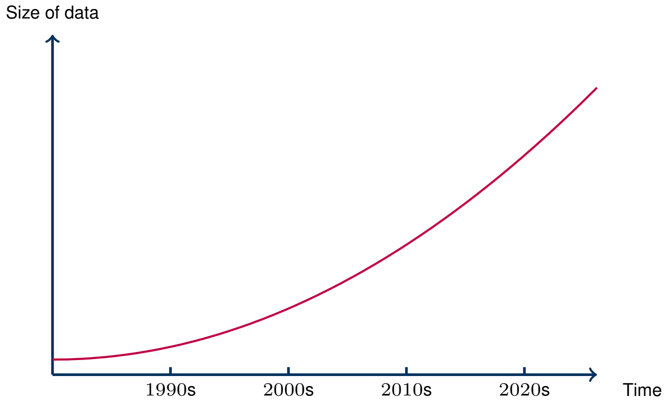
- The amount of data has been increased exponentially in the past;



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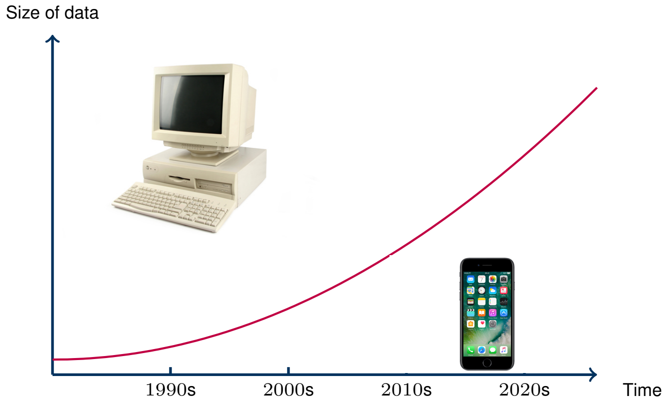

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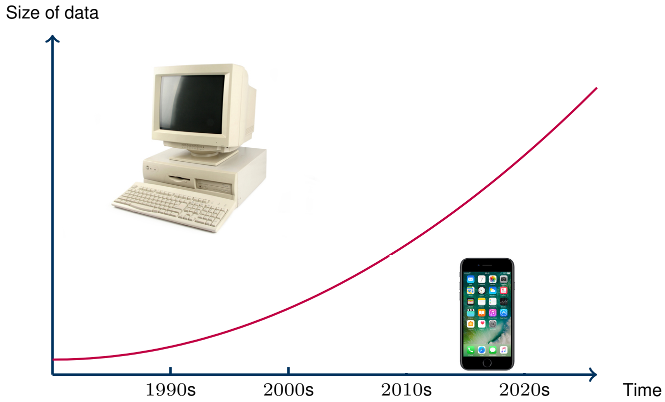
- For many applications computational devices' memories are limited;
- We only need good approximate solutions!



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- The input of a streaming algorithm is given as a data stream, which is a sequence of data
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(ε, δ)-APPROXIMATION

For confidence parameter ε and approximation parameter δ, the algorithm's output Output and the exact answer Exact satisfies
$$\mathbb{P}\left[\mathsf{Output} \in (1-\varepsilon, 1+\varepsilon) \cdot \mathsf{Exact}\right] \geq 1 - \delta.$$

Two models of streaming algorithms

Cash register model: every item in stream $\mathcal{S}$ is an item in U.

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Turnstile model: every item s_i in $\mathcal{S}$ associates with " +" or "-", which indicates if s_i is added into or deleted from $\mathcal{S}$.

- "+" indicates that s_i is added into the dataset;
- "-" indicates that s_i is deleted from the dataset.

Why turnstile model?

- Data may be added or deleted over time, e.g. Facebook graph.
- We need *robust* algorithms to handle this situation.

Outline

- Recall: Pairwise independent hashing
- AMS algorithm
- BJKST algorithm
- Chernoff Bound

Recall: Pairwise independent hash functions

PAIRWISE INDEPENDENCE

A family of functions $H = \{h \mid h : U \mapsto [n]\}$ is pairwise independent if, for any h chosen uniformly at random from H, the following holds:

1. $h(x)$ is uniformly distributed in $[n]$ for any $x \in U$;
2. For any $x_1 \neq x_2 \in U$, $h(x_1)$ and $h(x_2)$ are independent.

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THEOREM

Let p be a prime number, and let $h_{a,b}(x) = (ax + b) \bmod p$. Define

$$H = \{h_{a,b} \mid 0 \leq a, b \leq p - 1\}.$$

Then H is a family of pairwise independent hash functions.

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Norm Estimation: the Alon-Matias-Szegedy algorithm

F_p-NORM

Let U with $|U| = n$ be a dataset, and m_j be the number of occurrences of j in a stream. The F_p-norm is defined by

$$F_p \triangleq \sum_{i \in U} |m_i|^p \,.$$

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Alon, Matias, and Szegedy (1996) presented a systematical study for approximating the frequency moments.

- The numbers F_0, F_1, F_2 can be approximated in logarithmic space.
- Approximating F_k for $k \geq 6$ requires $n^{\Omega(1)}$ space.
- The paper won 2005 Gödel Award for “their foundational contribution to streaming algorithms”.

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- with probability $1/2$, we have $\rho(h(x)) = 1$

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- $\ldots$
- with probability $1/2^r$, we have $\rho(h(x)) = r$

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Proof of the theorem (1/2)

Let $X_{r,j}$ be a 0/1 indicator random variable such that

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Let p be the smallest index such that $2^{p+1/2} \geq 3F_0$. Then

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Let q be the largest index such that $2^{q+1/2} \leq F_0/3$. Then

$$\mathbb{P}[Z \leq F_0/3] = \mathbb{P}[z \leq q] = \mathbb{P}[Y_{q+1} = 0] \leq \frac{2^{q+1}}{F_0} \leq \frac{\sqrt{2}}{3}.$$

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    if ρ(h(x)) ≥ z, then
        B = B ∪ {(x, ρ(h(x)))}
        while |B| ≥ 100/ε^2
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THEOREM

The medium of the returned values from $\Theta(\log(1/\delta))$ independent copies of the algorithm above gives an (ε, δ)-approximation of F_0.

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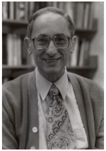
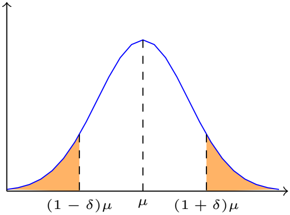
Chernoff Bounds

- Chernoffs bounds are "strong" bounds on the tail probabilities of sums of independent random variables (random variables can be discrete or continuous)
- usually these bounds decrease exponentially as opposed to a polynomial decrease in Markov's or Chebysheff's inequality (see example later)
- have found various applications in:
  - Approximation and Sampling Algorithms
  - Learning Theory (e.g., PAC-learning)
  - Statistics



Hermann Chernoff (1923-)



A simple Chernoff Bound for uniform coin flips

Uniform Chernoff Bound

Let $X_1, \ldots, X_n$ be independent random variables with $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$. Let $X := \sum_{i=1}^{n} X_i$. Then for any $\lambda > 0$,

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- This is a simple yet important setting, since r.v.'s are identical and symmetric.
- Bound on $\mathbb{P}[X \leq -\lambda]$ follows by symmetry.
- Bounds for the case $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = 0] = 1/2$ through substitution, see below.

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Corollary (Homework)

Let $Y_1, \ldots, Y_n$ be independent random variables with $\mathbb{P}[Y_i = 0] = \mathbb{P}[Y_i = 1] = 1/2$. Let $Y := \sum_{i=1}^{n} Y_i$ and $\mu := \mathbb{E}[Y] = n/2$. Then for any $0 < \lambda < \mu$,

$$\mathbb{P}[Y \geq \mu + \lambda] \leq \mathrm{e}^{-2\lambda^2/n}.$$

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$$\mathbb{P}[|X - \mu| \geq t] \leq \frac{\mathbb{V}[X]}{t^2},$$

  and plugging in $t = 25$ gives an upper bound of $25/25^2 = 1/25 = 0.04$, much better than what we obtained by Markov's inequality.

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- The uniform Chernoff bound (Corollary) with $\mu = 50, \lambda = 25$ gives:

$$\mathbb{P}[X \geq \mu + \lambda] \leq \mathrm{e}^{-2\lambda^2/100} = \mathrm{e}^{-625/50} = \mathrm{e}^{-12.5} = 0.00000372\ldots.$$

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Chernoff bound yields a more accurate result but needs independence!

Chernoff Bounds

Chernoff Bound (Multiplicative Version)

Let $X_1, \ldots, X_n$ be independent random variables with $\mathbb{P}[X_i = 1] = p_i$ and $\mathbb{P}[X_i = 0] = 1 - p_i$ for each i. Let $X := \sum_{i=1}^{n} X_i$ and $\mu := \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then:

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- For any $\varepsilon \geq 0$,

$$\mathbb{P}[X \geq (1+\varepsilon)\mu] \leq \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{(1+\varepsilon)}} \right)^{\mu}.$$

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- For any $\varepsilon \in [0,1]$, the inequality can be simplified to

$$\mathbb{P}[|X - \mu| \geq \varepsilon\mu] \leq 2e^{-\mu\varepsilon^2/3}.$$