University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Solutions 3

Solutions

Problem 1: Project the volume of a d-dimensional ball of radius $\sqrt{d}$ onto a line through the centre. For large d, give an intuitive argument that the projected volume should behave like a Gaussian.

Solution: Projecting the volume of a d-dimensional ball corresponds to assigning a function $f : [-\sqrt{d}, \sqrt{d}]$ such that $f(x)$ is the volume of a $(d-1)$-dimensional ball with radius $\sqrt{d-x^2}$. This function is 0 on the endpoints of its domain, reaches its maximal value for $x = 0$ and is an even function, i.e., $f(-x) = f(x)$. We will show that, as x is increasing on the interval $[0, \sqrt{d}]$, the function f decreases exponentially in x^2.

We will assume that $d - 1 = 2k$, for some natural number k, as the other case is done similarly. Let $V_n(R)$ denote the volume of a ball in n dimensions of radius R. Then, we know that

$$\begin{aligned} f(x) = V_{d-1}\left(\sqrt{d-x^2}\right) &= \frac{\pi^k}{k!} \cdot \left(\sqrt{d-x^2}\right)^{d-1} \\ &= \frac{\pi^k}{k!} \cdot d^{(d-1)/2} \cdot \left(1 - \frac{x^2}{d}\right)^{(d-1)/2} \\ &\approx \frac{\pi^k}{k!} \cdot d^{(d-1)/2} \cdot \mathrm{e}^{-\frac{x^2}{2}}. \end{aligned}$$

Problem 2: Consider a nonorthogonal basis $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_d}$. The $\mathbf{e_i}$ are a set of linearly independent unit vectors that span the space.

1. Prove that the representation of any vector in this basis is unique;
2. Calculate the squared length of $\mathbf{z} = \frac{\sqrt{2}}{2}\mathbf{e_1} + \mathbf{e_2}$ where $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
3. If $\mathbf{y} = \sum_{i=1}^{d} a_i\mathbf{e_i}$ and $\mathbf{z} = \sum_{i=1}^{d} b_i\mathbf{e_i}$, with $0 < a_i < b_i$ for all $1 \le i \le d$, is it necessarily true that the length of $\mathbf{z}$ is greater than the length of $\mathbf{y}$? If yes give a proof of the statement, if no find a counterexample;
4. Consider the basis $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.
   (a) What is the representation of the vector $\mathbf{v} = (0, 1)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$? I.e. find scalars a, b such that $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2}$.
   (b) What is the representation of the vector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$?
   (c) What is the representation of the vector $(1, 2)$ in the basis $(\mathbf{e_1}, \mathbf{e_2})$?

Solution:

1. Let $\mathbf{v}$ be an arbitrary vector. Since the vectors $\{\mathbf{e_i}\}$'s form a basis, they span the entire space. So there exists a representation $v = \sum_{i=1}^{d} \alpha_i\mathbf{e_i}$. Suppose $\mathbf{v}$ can also be represented as $\sum_{i=1}^{d} \beta_i\mathbf{e_i}$. Then it holds that
$$\sum_{i=1}^{d} \alpha_i\mathbf{e_i} = \sum_{i=1}^{d} \beta_i\mathbf{e_i}$$
which gives us that $\sum_{i=1}^{d}(\alpha_i - \beta_i)\mathbf{e_i} = 0$. Since the vectors are linearly independent, we must have that $\alpha_i = \beta_i$ for all i. We conclude that the two representations are the same.

2. We have $\mathbf{z} = \frac{\sqrt{2}}{2}\mathbf{e_1} + \mathbf{e_2} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$. Therefore the length of $\mathbf{z}$ is $\|\mathbf{z}\| = \frac{\sqrt{2}}{2}$.

3. We will show that the answer is false. Let $\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_d}$ be the standard basis, i.e. $\mathbf{u_i}$ is the vector with 1 in position i and 0 everywhere else. We can rewrite the vectors $\mathbf{e_i}$, for all i, as follows:

$$\mathbf{e_i} = \sum_{j=1}^{d} \alpha_{ij}\mathbf{u_j}.$$

Substituting in the representations of $\mathbf{y}$ and $\mathbf{z}$ we see that

$$\mathbf{y} = \sum_{i=1}^{d} a_i \left(\sum_{j=1}^{d} \alpha_{ij}\mathbf{u_j} \right) = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} a_i\alpha_{ij} \right) \mathbf{u_j},$$
$$\mathbf{z} = \sum_{i=1}^{d} b_i \left(\sum_{j=1}^{d} \alpha_{ij}\mathbf{u_j} \right) = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} b_i\alpha_{ij} \right) \mathbf{u_j}.$$

Moreover, we see that the two norms can be expressed as

$$\|y\|^2 = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} a_i\alpha_{ij} \right)^2$$
$$\|z\|^2 = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} b_i\alpha_{ij} \right)^2.$$

Since the two lengths are positive real numbers, it is sufficient to compare their squared norms. We have that

$$\|z\|^2 - \|y\|^2 = \sum_{j=1}^{d} \left(\left(\sum_{i=1}^{d} b_i\alpha_{ij} \right)^2 - \left(\sum_{i=1}^{d} a_i\alpha_{ij} \right)^2 \right) = \sum_{j=1}^{d} \left(\left(\sum_{i=1}^{d} \alpha_{ij}(a_i + b_i) \right) \left(\sum_{i=1}^{d} \alpha_{ij}(b_i - a_i) \right) \right).$$

For certain choices of the numbers α_{ij}, $0 < a_i < b_i$, the right hand side can be negative. For example, take

$$\alpha = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}, \quad a_1 = 0.9, \quad a_2 = 0.1, \quad b_1 = b_2 = 1.$$

4. Suppose $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2}$. We substitute the values for the three vectors and solve for a and b as follows:

   (a)
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \Leftrightarrow \begin{cases} a - \frac{\sqrt{2}}{2}b & = 0 \\ \frac{\sqrt{2}}{2}b & = 1 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = \sqrt{2} \end{cases}$$

   (b) $a = \sqrt{2}$, $b = 1$.

   (c) $a = 3$, $b = 2\sqrt{2}$.

Problem 3: Compute the right-singular vectors v_i, the left-singular vectors u_i, the singular values σ_i and hence find the *Singular value decomposition* of

1. $A = \begin{pmatrix} 1 & 1 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}$;

2. $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 3 \\ 3 & 1 \end{pmatrix}$.

Solution: Throughout the solution we will make use of the following lemma.

Lemma 1. *Let a and b be two real numbers satisfying $a^2 + b^2 = 1$ and $a \geq 0$. The product ab is maximised when $a = b = \frac{\sqrt{2}}{2}$.*

Proof. Using the initial conditions we can rewrite $a = \sqrt{1-b^2}$. Hence maximising the product ab reduces to maximising the function $f(x) = x\sqrt{1-x^2}$. A point x_0 maximises $f(x)$ if $x_0 \geq 0$ and $f'(x_0) = 0$. We have that

$$f'(x) = \sqrt{1-x^2} + \frac{-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.$$

We conclude that $x_0 = \frac{\sqrt{2}}{2}$ which gives $a = b = \frac{\sqrt{2}}{2}$. □

1. For finding the first right-singular vector v_1, we look at any vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $\|v\| = 1$ and v maximises $\|Av\|$. Without loss of generality we can also assume that $a \geq 0$. Firstly, note that maximising $\|Av\|$ is equivalent to maximising $\|Av\|^2$. We also have that:

$$\|Av\|^2 = \left\| \begin{pmatrix} a+b \\ 3b \\ 3a \end{pmatrix} \right\|^2 = (a+b)^2 + 9b^2 + 9a^2.$$

Since $\|v\| = 1$, we have that $a^2 + b^2 = 1$. Therefore $\|Av\|^2 = 10(a^2+b^2) + 2ab = 10 + 2ab$. We see that $\|Av\|^2$ is maximised if and only if ab is maximised. Using Lemma 1 that happens when $a = b = \frac{1}{\sqrt{2}}$. So the first right-singular vector $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the first singular value is $\sigma_1 = \|Av_1\| = \sqrt{11}$. For the first left-singular vector u_1 we compute

$$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{22}} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}.$$

For the second right-singular vector v_2, we look at vectors $v = \begin{pmatrix} a' \\ b' \end{pmatrix}$ such that $\|v\| = 1$, $v \perp v_1$ and v maximises $\|Av\|$. Without loss of generality we can assume $a' \geq 0$. Since $v \perp v_1$ this implies that $a' + b' = 0$. Solving $a'^2 + b'^2 = 1$ gives us that $a' = \frac{1}{\sqrt{2}}$. Hence $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Moreover, the second singular value is $\sigma_2 = \|Av_2\| = 3$. The second left-singular vector u_2 is computed by

$$u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The singular value decomposition of A is

$$A = UDV^T = \begin{pmatrix} \frac{2}{\sqrt{22}} & 0 \\ \frac{3}{\sqrt{22}} & \frac{-1}{\sqrt{2}} \\ \frac{3}{\sqrt{22}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{11} & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

2. Again, for finding v_1 we look at any vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $\|v\| = 1$ and v maximises $\|Av\|$. Without loss of generality we can assume $a \geq 0$.

$$\|Av\|^2 = \left\| \begin{pmatrix} 2b \\ 2a \\ a+3b \\ 3a+3b \end{pmatrix} \right\|^2 = 4a^2 + 4b^2 + (a+3b)^2 + (3a+b)^2 = 14(a^2+b^2) + 12ab.$$

Using that $\|v\| = 1$ we have that $\|Av\|^2 = 14 + 12ab$ which, by Lemma 1, is maximised for $a = b = \frac{1}{\sqrt{2}}$. Therefore we have that $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\sigma_1 = \sqrt{20}$. The first left-singular vector u_1 is given by

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{2\sqrt{10}} \begin{pmatrix} 2 \\ 2 \\ 4 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}.$$

A similar reasoning to the previous part tells us that $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\sigma_2 = \|Av_2\| = \sqrt{8}$. We also have that

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{4} \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Hence, the singular value decomposition of A is

$$A = UDV^T = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{\sqrt{10}} & \frac{1}{2} \\ \frac{2}{\sqrt{10}} & -\frac{1}{2} \\ \frac{2}{\sqrt{10}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{8} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$