University of Edinburgh

INFR11156: Algorithmic Foundations of Data Science (2019)

Solution 9

Problem 1: Let $G = (V, E)$ be an arbitrary undirected graph with n vertices.

1. Assume that H_1 and H_2 are $(1+\varepsilon)$-spectral sparsifiers of G with corresponding Laplacian matrices L_{H_1} and L_{H_2}. Prove or disprove by counterexample the following statement: "The graph defined by the Laplacian matrix $(1/2) \cdot (L_{H_1} + L_{H_2})$ is a $(1 + \varepsilon)$-spectral sparsifier of G as well."

2. Assume that $G = (V, E)$ is an unweighted and complete graph. Prove or disprove the following statement: "The graph consisting of $O(n \log n/\varepsilon^2)$ edges, where every edge is sampled uniformly at random from G, is a $(1 + \varepsilon)$-spectral sparsifier of G. "

3. Assume that $G = (V, E)$ is an arbitrary unweighted graph. Prove or disprove the following statement: "The graph consisting of $O(n \log n/\varepsilon^2)$ edges, where every edge is sampled uniformly at random from G, is a $(1 + \varepsilon)$-spectral sparsifier of G. "

4. Prove or disprove by counterexample the following statement: "If H is a $(1 + \varepsilon)$-spectral sparsifier of $G = (V, E)$, then for any subset $S \subseteq V$, the cut value between S and $V \setminus S$ in G and the one in H is approximately the same up to a multiplicative factor of $(1 \pm \varepsilon)$."

Solution:

1. By definition of $(1 + \varepsilon)$-spectral sparsifiers we have that for any $x \in \mathbb{R}^n$ it holds

$$(1 - \varepsilon)x^\intercal L_G x \leq x^\intercal L_{H_1} x \leq (1 + \varepsilon)x^\intercal L_G x$$
$$(1 - \varepsilon)x^\intercal L_G x \leq x^\intercal L_{H_2} x \leq (1 + \varepsilon)x^\intercal L_G x$$

   Adding the two equations and multiplying by 1/2 implies that the graph defined by the Laplacian matrix $(1/2) \cdot (L_{H_1} + L_{H_2})$ is a $(1 + \varepsilon)$-spectral sparsifier of G.

2. First, we will show that in a complete graph the effective resistance of every edge e, $\mathrm{Reff}(e)$ is the same. To see this, recall that the effective resistance $\mathrm{Reff}(e)$ in an undirected graph G is the probability that e is selected in a spanning tree of G. However, by symmetry, these probabilities must be the same for every edge e. Note that since the graph is unweighted, the leverage coefficients $\ell_e = \mathrm{Reff}(e)$.

   Since the leverage coefficients are constant, we can essentially follow the algorithm presented in the lecture notes for constructing spectral sparsifiers, noting that the sampling probability for every edge is the same. This corresponds to sampling the edges uniformly at random from $E(G)$. Hence, after reweighing all the sampled edges by the same weight, this method indeed produces a $(1 + \varepsilon)$ spectral sparsifier of G.

3. The statement is incorrect, and the following graph G can be used as a counterexample: let G be the graph consisting of 2 disjoint complete graphs, each of which has $n/2$ vertices, and a single edge connecting these two graphs. Hence, graph G has $\Omega(n^2)$ edges and, the probability that the middle edge doesn't get sampled is

$$1 - \Theta\left(\frac{n \log n}{n^2 \cdot \varepsilon^2}\right) = 1 - \Theta\left(\frac{\log n}{n \cdot \varepsilon^2}\right).$$

   On the other side, as long as the middle edge isn't sampled, the resulting graph is disconnected and cannot form a spectral sparsifier of G.

4. Let S be a subset of V and χ_S be the indicator vector of the set S. Then, it holds that

$$\chi_S^\intercal L_G \chi_S = \sum_{u \sim v} w(u,v) \left(\chi_S(u) - \chi_S(v)\right)^2 = w_G(S, V/S),$$

and similarly

$$\chi_S^\intercal L_H \chi_S = w_H(S, V/S).$$

By definition of spectral sparsifier, we see that

$$(1-\varepsilon) w_G(S, V/S) \le w_H(S, V/S) \le (1+\varepsilon) w_G(S, V/S).$$

Problem 2: Let H be a $(1+\varepsilon)$-spectral sparsifier of graph $G = (V, E, w)$ for some constant $\varepsilon \in (0, 1/3)$. Prove that, for any set $S \subset V$, the conductance $\phi_H(S)$ of set S in H and the one $\phi_G(S)$ satisfies

$$\phi_H(S) \le (1+3\varepsilon)\phi_G(S).$$

Solution: Let $S \subset V$ be an arbitrary set, and let χ_S be the indicator vector of S. Then, it holds that

$$\begin{aligned}
\frac{\phi_H(S)}{\phi_G(S)} &= \frac{\chi_S^\intercal L_H \chi_S}{\chi_S^\intercal L_G \chi_S} \cdot \frac{\mathrm{vol}_G(S)}{\mathrm{vol}_H(S)} \\
&\le (1+\varepsilon) \cdot \frac{\sum_u w_G(u)}{\sum_u w_H(u)} \\
&= (1+\varepsilon) \cdot \frac{\sum_u \chi_u^\intercal L_G \chi_u}{\sum_u \chi_u^\intercal L_H \chi_u} \\
&\le (1+\varepsilon) \cdot \frac{1}{1-\varepsilon} \cdot \frac{\sum_u \chi_u^\intercal L_H \chi_u}{\sum_u \chi_u^\intercal L_H \chi_u} \\
&\le (1+3\varepsilon),
\end{aligned}$$

where the inequalities in lines 2 and 4 follow from the definition of $(1+\varepsilon)$-spectral sparsification and the inequality in the last line from the choice of $\varepsilon \in (0, 1/3)$.