INFR10086 Machine Learning (MLG) Semester 2, 2023/4

Analytic Geometry

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Definition 1. A vector is an element in $\mathbb{R}^d$ that consists of d coordinates. For a vector x, we write its d coordinates as $(x_1, x_2, \dots, x_d)$.

Definition 2. The multiplication of a vector x by a scalar $a \in \mathbb{R}$ and the addition of two vectors u and v are defined as

- $ax = (ax_1, ax_2, \dots, ax_d)$
- $u + v = (u_1 + v_1, \dots, u_d + v_d)$

- The notation $u - v$ is a more convenient way of writing $u + (-1)v$.

Definition 3. The dot product between two vectors u and v is defined as

$$u^\top v = u_1 v_1 + \cdots + u_d v_d = \sum_{i=1}^{d} u_i v_i. \tag{1}$$

Example 1. The dot product satisfies the following properties.

- bilinearity
  - $(au)^\top v = a(u^\top v) = u^\top (av)$ for any two vectors u and v and a scalar $a \in \mathbb{R}$.
  - $(u + v)^\top w = u^\top w + v^\top w$ for any three vectors u, v, and w.
  - $w^\top (u + v) = w^\top u + w^\top v$ for any three vectors u, v, and w.
- commutativity
  - $u^\top v = v^\top u$ for any two vectors u and v.

- These properties defines a inner product. We can show that dot product satifies all the above properties and, hence, is an inner product.
- The inner product of u and v is written as $\langle u, v \rangle$. The notation $u^\top v$ is only for the dot product.
- The proofs of these properties are left as exercises.

Definition 4. The ℓ_2 norm of a vector v is defined as

$$\|v\|_2 = \sqrt{v^\top v} = \sqrt{v_1^2 + \cdots + v_d^2}. \tag{2}$$

- There are other norms. For example, the ℓ_1 norm $\|v\|_1 = \sum_{i=1}^d |v_i|$.

Example 2. In a two-dimensional space, a vector $x = (x_1, x_2)$ has an ℓ_2 norm $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$. Intuitively, the ℓ_2 norm is the length of a vector, and this is true even in d-dimensional spaces.

Example 3. The ℓ_2 norm satisfies the following properties

- nonnegativity
  - $\|u\| \geq 0$ for any vector u.
- $\|au\| = |a|\|u\|$ for any scalar $a \in \mathbb{R}$ and a vector u
- $\|u\| = 0$ only if the vector u is 0.
- triangle inequality
  - $\|u\| + \|v\| \geq \|u + v\|$ for any two vectors u and v.

- These properties defines a norm, so the subscript 2 is intentionally left out. We can show that the ℓ_2 norm satisfies the above properties and, hence, is a norm.
- In fact, if we have an inner product, we can show that $\sqrt{\langle x, x\rangle}$ is a norm. We can conveniently write $\|x\| = \sqrt{\langle x, x\rangle}$.
- The proofs of these properties are left as exercises.

Example 4. Show that

$$\|u + v\|_2^2 = \|u\|_2^2 + 2u^\top v + \|v\|_2^2. \tag{3}$$

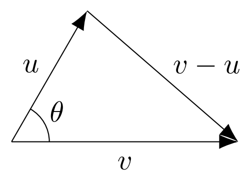
$$\begin{aligned}
\|u + v\|_2^2 &= (u + v)^\top (u + v) && \text{definition of } \ell_2 \text{ norm} && (4)\\
&= (u + v)^\top u + (u + v)^\top v && \text{bilinearity} && (5)\\
&= u^\top u + v^\top u + u^\top v + v^\top v && \text{bilinearity} && (6)\\
&= \|u\|_2^2 + 2u^\top v + \|v\|_2^2 && \text{commutativity and definition of } \ell_2 \text{ norm} && (7)
\end{aligned}$$

Example 5. Show that $v/\|v\|_2$ has an ℓ_2 norm of 1 for any vector $v \neq 0$.

$$\left\| \frac{v}{\|v\|_2} \right\|_2 = \frac{1}{\|v\|_2} \|v\|_2 = 1. \tag{8}$$

Example 6. If two vectors u and v have an angle θ between them, show that

$$\cos\theta = \frac{u^\top v}{\|u\|_2 \|v\|_2}. \tag{9}$$



Based on the law of cosine (the figure above), we have

$$\|v - u\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - 2\|u\|_2\|v\|_2 \cos\theta. \tag{10}$$

We can expand the left hand side into

$$\|v - u\|_2^2 = \|v\|_2^2 - 2u^\top v + \|u\|_2^2. \tag{11}$$

By comparing the two equations, we have the desired result.

Example 7. Show that

$$|\langle u, v\rangle| \le \|u\|\|v\|. \tag{12}$$

This is known as the Cauchy–Schwarz inequality.

- The strategy of the proof is to complete the square.

$$\|u\|^2\|v\|^2 - (\langle u, v\rangle)^2 = \|u\|^2 \left(\|v\|^2 - \frac{(\langle u, v\rangle)(\langle u, v\rangle)}{\|u\|^2} \right) \tag{13}$$

$$= \|u\|^2 \left[\|v\|^2 - 2\left\langle \frac{\langle u, v\rangle}{\|u\|^2} u, v \right\rangle + \left\langle \frac{\langle u, v\rangle}{\|u\|^2} u, \frac{\langle u, v\rangle}{\|u\|^2} u \right\rangle \right] \tag{14}$$

$$= \|u\|^2 \left\| v - \frac{\langle u, v\rangle}{\|u\|^2} u \right\|^2 \ge 0 \tag{15}$$

- The proof might seem ingenious, but there is actually a geometric interpretation that we will see later.

Example 8. Show that

$$-1 \le \frac{u^\top v}{\|u\|_2\|v\|_2} \le 1 \tag{16}$$

when both u and v are not 0.

- By Cauchy–Schwarz, $(u^\top v)^2 \leq \|u\|_2^2 \|v\|_2^2$ implies

$$\left(\frac{u^\top v}{\|u\|_2\|v\|_2}\right)^2 \leq 1. \tag{17}$$

Definition 5. A line is a set of points $\{x : x = u + tv \text{ for } t \in \mathbb{R}\}$ for any vector u and vector $v \neq 0$.

- Think of a line as shooting a ray along the direction of v from the point u.

Example 9. In two dimensions, a line consists of points (x, y) that satisfies

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{18}$$

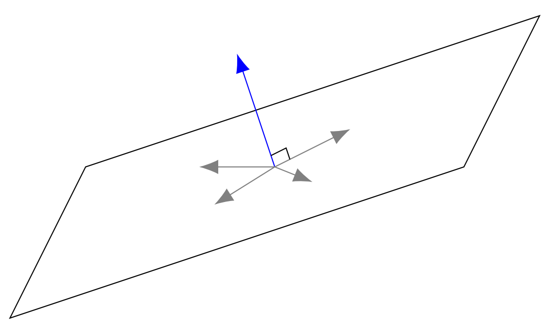
If we write

$$y = u_2 + tv_2 = u_2 + \frac{x - u_1}{v_1} v_2 = \underbrace{\frac{v_2}{v_1}}_{a} x + \underbrace{\left(u_2 - \frac{v_2}{v_1}\right)}_{b}, \tag{19}$$

we have $y = ax + b$, the familiar way of writing a line in 2D.

Definition 6. A plane is a set of points $\{x : v^\top (x - u) = 0\}$ for any vector u and vector $v \neq 0$.

The vector v is the normal vector perpendicular to the plane, and u is a shift from the origin.



Example 10. In three dimensions, a plane consists of points (x, y, z) that satisfies

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^\top \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) = 0. \tag{20}$$

If we write

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}^\top \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) = \underbrace{v_1}_{a} x + \underbrace{v_2}_{b} y + \underbrace{v_3}_{c} z + \underbrace{(-v_1 u_1 - v_2 u_2 - v_3 u_3)}_{d} = 0, \tag{21}$$

we get $ax+by+cz+d = 0$, the familiar way of writing a plane in 3D. Note that $(a, b, c) = (v_1, v_2, v_3)$ is the normal vector.

Example 11. Show that points that satisfies $y = w^\top x + b$, where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$, forms a plane.

We can rewrite $y = w^\top x + b$ as

$$\begin{pmatrix} w \\ -1 \end{pmatrix}^\top \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix} \right) = 0. \tag{22}$$

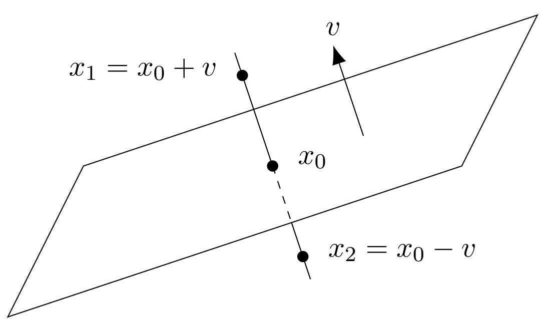
The points $(x, y) \in \mathbb{R}^{d+1}$ form a plane, with a normal vector $(w, -1) \in \mathbb{R}^{d+1}$ and a shift $(0, b) \in \mathbb{R}^{d+1}$.

Example 12. Show that u is on the plane $\{x : v^\top (x - u) = 0\}$.

The point u is on the plane because $v^\top (u - u) = 0$.

Example 13. Show that points $\{x : v^\top (x - u) > 0\}$ and $\{x : v^\top (x - u) < 0\}$ belong to two sides of the plane $\{x : v^\top (x - u) = 0\}$.

- To get to either side of the plane, we first start from a point x_0 on the plane and move along the line parallel the normal vector v.



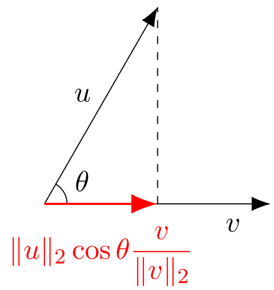
The fact that x_0 is on the plane means that $v^\top (x_0 - u) = 0$. The new point $x_0 + tv$ would satisfy

$$v^\top ((x_0 + tv) - u) = v^\top (x_0 - u) + t\|v\|_2^2 = t\|v\|_2^2. \tag{23}$$

The above expression is positive if $t > 0$. All points with $t > 0$ are on one side of the plane and, by the above, satisfy $\{x : v^\top (x - u) > 0\}$. Similarly, all points with $t < 0$ are on the other side of the plane, satisfying $\{x : v^\top (x - u) < 0\}$.

- Points are one side of a plane constitute a halfspace.

Definition 7. The vector $\|u\|_2 \cos\theta \frac{v}{\|v\|_2}$ is a projection of u on v, where u and v has an angle θ.



Projection is the act of casting a shadow. It's not hard to see that $|\|u\|_2 \cos\theta|$ is the length of u projected on v. To give it a direction, we multiply $\|u\|_2 \cos\theta$ by $v/\|v\|_2$. Note that $v/\|v\|_2$ offers a direction and has an ℓ_2 norm of 1.

Example 14. The projection of u on v is

$$\frac{u^\top v}{\|v\|_2} \frac{v}{\|v\|_2}. \tag{24}$$

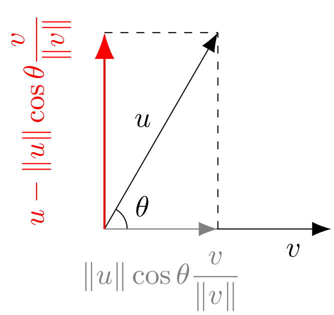
$$\|u\|_2 \cos\theta \frac{v}{\|v\|_2} = \|u\|_2 \frac{u^\top v}{\|u\|_2 \|v\|_2} \frac{v}{\|v\|_2} \frac{v}{\|v\|_2} = \frac{u^\top v}{\|v\|} \frac{v}{\|v\|} \tag{25}$$

Example 15. The vector

$$u - \frac{u^\top v}{\|v\|_2} \frac{v}{\|v\|_2} \tag{26}$$

is perpendicular to the projection of u on v.

- Intuitively, we have the figure below.



We can verify this algebraically.

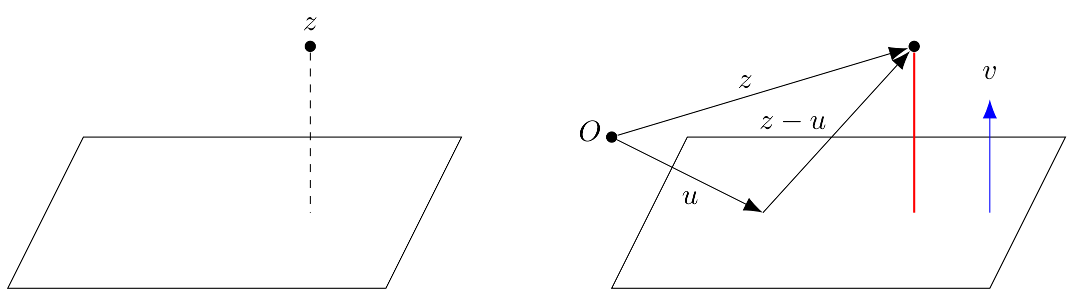
$$\left(\frac{u^\top v}{\|v\|_2}\frac{v}{\|v\|_2}\right)^\top\left(u-\frac{u^\top v}{\|v\|_2}\frac{v}{\|v\|_2}\right)=\left(\frac{u^\top v}{\|v\|_2}\right)^2-\left(\frac{u^\top v}{\|v\|_2}\right)^2=0 \tag{27}$$

- The perpendicular part is the term that appears in the proof of Cauchy–Schwarz. In other words, the proof Cauchy–Schwarz requires that the perpendicular part has a non-zero norm, and this is true as any norm of a vector is non-zero.

Example 16. The distance between a point z and a plane $\{x : v^\top(x-u)=0\}$ is

$$\frac{|v^\top(z-u)|}{\|v\|_2}. \tag{28}$$

- The strategy is to project $z-u$ on the normal vector v. Note the shift u.



- The projection of $z-u$ on v is

$$\frac{v^\top(z-u)}{\|v\|_2}\frac{v}{\|v\|_2}. \tag{29}$$

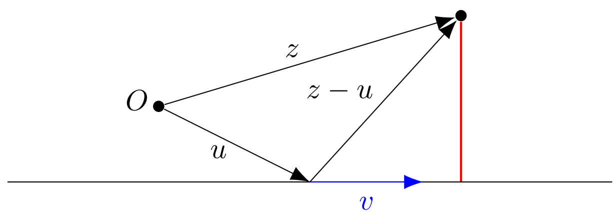
The norm of this vector is

$$\frac{|v^\top(z-u)|}{\|v\|_2}. \tag{30}$$

Example 17. The distance between a point z and a line $\{x : x=u+tv \text{ for any } t\in\mathbb{R}\}$ is

$$\left\|(z-u)-\frac{v^\top(z-u)}{\|v\|_2}\frac{v}{\|v\|_2}\right\|_2. \tag{31}$$

- The strategy is to project $z-u$ on the line, i.e., the vector v, and to get the perpendicular part.



- The perpendicular part after the projection is

$$(z-u)-\frac{v^\top(z-u)}{\|v\|_2}\frac{v}{\|v\|_2}. \tag{32}$$

The distance is the length of that vector, i.e.,

$$\left\|(z-u)-\frac{v^\top(z-u)}{\|v\|_2}\frac{v}{\|v\|_2}\right\|_2. \tag{33}$$