Machine Learning: Analytic Geometry

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Vectors in $\mathbb{R}^d$

- $x = (x_1, x_2, \ldots, x_d)$
- $ax = (ax_1, ax_2, \ldots, ax_d)$ for $a \in \mathbb{R}$
- $u + v = (u_1 + v_1, \ldots, u_d + v_d)$
- $u - v = u + (-1)v$

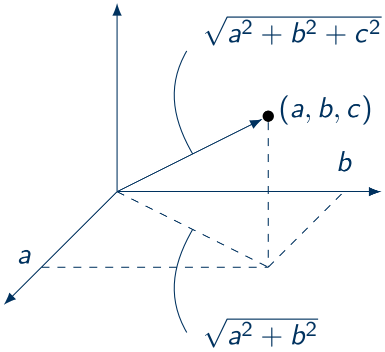
Dot product

- $u^\top v = u_1 v_1 + \cdots + u_d v_d = \sum_{i=1}^{d} u_i v_i$
- $(au)^\top v = a(u^\top v) = u^\top (av)$ for $a \in \mathbb{R}$
- $(u + v)^\top w = u^\top w + v^\top w$
- $w^\top (u + v) = w^\top u + w^\top u$
- $u^\top v = v^\top u$

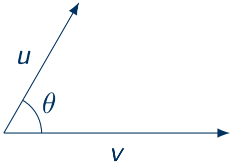
The ℓ_2 norm

- $\|v\|_2 = \sqrt{v^\top v} = \sqrt{v_1^2 + \cdots + v_d^2}$
- $\|au\|_2 = |a|\|u\|_2$ for $a \in \mathbb{R}$
- $\|u\|_2 \geq 0$
- If $\|u\|_2 = 0$, then $u = 0$
- $\|u\| + \|v\| \geq \|u + v\|$

The ℓ_2 norm and length

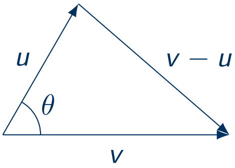


Dot product and angle



$$\cos\theta = \frac{u^\top v}{\|u\|_2 \|v\|_2} \tag{1}$$

Dot product and angle



By the law of cosine,

$$\|v - u\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - \|u\|_2 \|v\|_2 \cos\theta. \tag{2}$$

Comparing the above with

$$\|v - u\|_2^2 = \|v\|_2^2 - 2u^\top v + \|u\|_2^2, \tag{3}$$

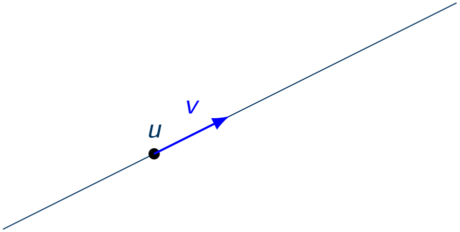
we get

$$\cos\theta = \frac{u^\top v}{\|u\|_2 \|v\|_2}. \tag{4}$$

A line

A line

A line is a set of points $\{x : x = u + tv \text{ for } t \in \mathbb{R}\}$ for any vector u and vector $v \neq 0$.



A line in 2D

- Is $y = ax + b$ a line?

A line in 2D

- Is $y = ax + b$ a line?

- We can rewrite $y = ax + b$ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} + t \begin{pmatrix} 1 \\ a \end{pmatrix}. \tag{5}$$

A plane

A plane

A plane (or hyperplane) is a set of points $\{x : v^\top(x - u) = 0\}$ for any vector u and vector $v \neq 0$.

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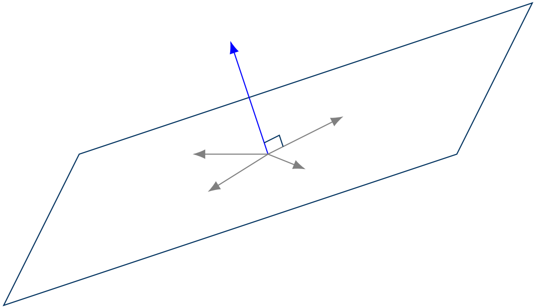


A plane in 3D

- Is $ax + by + cz + d = 0$ a plane?

A plane in 3D

- Is $ax + by + cz + d = 0$ a plane?
- We can rewrite $ax + by + cz + d = 0$ as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^{\top} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ d/c \end{pmatrix} \right) = 0. \tag{6}$$

A plane in $d + 1$ dimension

- Is $y = w^\top x + b$, where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$, a plane?

A plane in $d + 1$ dimension

- Is $y = w^\top x + b$, where $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$, a plane?
- We can rewrite $y = w^\top x + b$ as

$$\begin{pmatrix} w \\ -1 \end{pmatrix}^\top \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix} \right) = 0. \tag{7}$$

Halfspaces

- Points $\{x : v^\top(x-u) > 0\}$ are on one side of the plane $\{x : v^\top(x-u) = 0\}$.

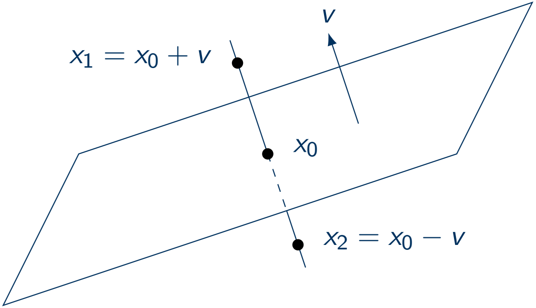
Halfspaces

- Points $\{x : v^\top(x-u) > 0\}$ are on one side of the plane $\{x : v^\top(x-u) = 0\}$.
- Consider the line $\{x : x = x_0 + tv \text{ for } t \in \mathbb{R}\}$ for some x_0 on the plane.

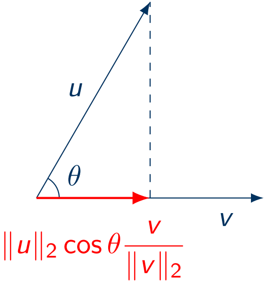
$$v^\top(x_0 + tv - u) = v^\top(x_0 - u) + t\|v\|_2^2 = t\|v\|_2^2 \tag{8}$$

When $t > 0$, $v^\top(x_0 + tv - u) > 0$.

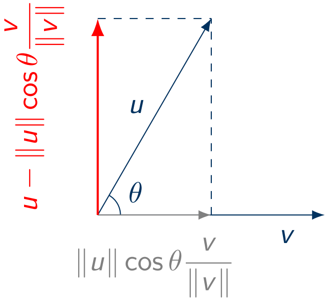
Halfspaces



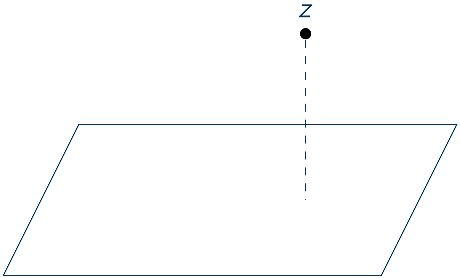
Projection



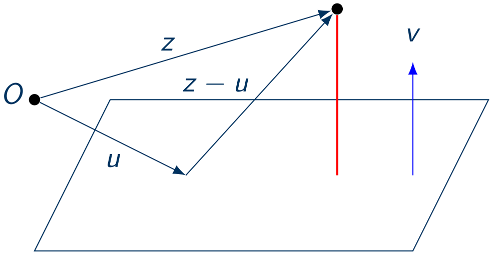
The perpendicular part after projection



The distance between a plane and a point



The distance between a plane and a point



The distance between a plane and a point

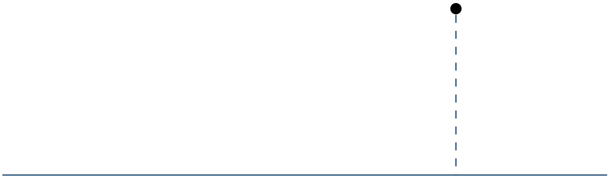
The projection of $z - u$ on v is

$$\frac{v^\top (z-u)}{\|v\|_2} \frac{v}{\|v\|_2}, \tag{9}$$

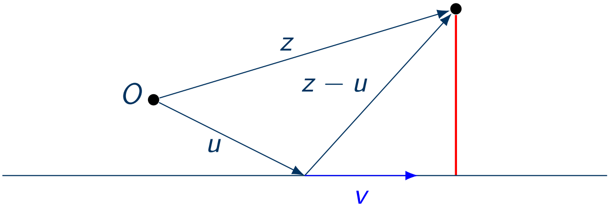
which has an ℓ_2 norm of

$$\frac{|v^\top (z-u)|}{\|v\|_2}. \tag{10}$$

The distance between a line and a point



The distance between a line and a point



The distance between a line and a point

The perpendicular part after the projection of $z - u$ on v is

$$(z - u) - \frac{v^\top (z - u)}{\|v\|_2} \frac{v}{\|v\|_2}, \tag{11}$$

with a norm of

$$\left\| (z - u) - \frac{v^\top (z - u)}{\|v\|_2} \frac{v}{\|v\|_2} \right\|_2 \tag{12}$$

Cauchy–Schwarz inequality

For any vector u and vector v,

$$(u^\top v)^2 \leq \|u\|_2^2 \|v\|_2^2. \tag{13}$$

Cauchy–Schwarz inequality

$$\|u\|_2^2\|v\|_2^2 - (u^\top v)^2 = \|u\|_2^2 \left(\|v\|_2^2 - \frac{(u^\top v)(u^\top v)}{\|u\|_2^2} \right) \tag{14}$$

$$= \|u\|_2^2 \left[\|v\|_2^2 - 2 \left(\frac{u^\top v}{\|u\|_2^2} u \right)^\top v + \left(\frac{u^\top v}{\|u\|_2^2} u \right)^\top \left(\frac{u^\top v}{\|u\|_2^2} u \right) \right] \tag{15}$$

$$= \|u\|_2^2 \left\| v - \frac{u^\top v}{\|u\|_2^2} u \right\|^2 \geq 0 \tag{16}$$