Notes on unknown uniform distributions in hierarchical probabilistic models

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A common situation within a probabilistic model is that some variables $\mathbf{x} = \{x_1, \ldots, x_N\}$ are assumed to have come from an unknown distribution. For large datasets it may be necessary to introduce a flexible or non-parametric prior over possible distributions. A simple assumption, often good enough for small N, is to assume that the $\{x_n\}$ came from a uniform distribution, $\mathrm{Uniform}[a, b]$, with a and b unknown. This note contains the marginal likelihood of this model for quick reference.

We put a broad uniform distribution on the unknown (a, b) bounds:

$$p(a, b) = \frac{2}{(B - A)^2}\, \mathbb{I}(b > a)\, \mathbb{I}(b < B)\, \mathbb{I}(a > A). \tag{1}$$

The above probability and all that follow are implicitly conditioned on hyper-parameters A and B. The likelihood of the parameters is:

$$p(\mathbf{x}|a, b) = \frac{1}{(b - a)^N}\, \mathbb{I}(x_{\max} < b)\, \mathbb{I}(x_{\min} > a). \tag{2}$$

The marginal likelihood is

$$\begin{aligned} p(\mathbf{x}) &= \int \mathrm{d}a \int \mathrm{d}b\;\; p(\mathbf{x}|a, b)\, p(a, b) \\ &= \frac{2}{(B - A)^2} \int_A^{x_{\min}} \mathrm{d}a \int_{x_{\max}}^{B} \mathrm{d}b\;\; \frac{1}{(b - a)^N} \\ &= \frac{(x_{\max} - x_{\min})^{2-N} + (B - A)^{2-N} - (B - x_{\min})^{2-N} - (x_{\max} - A)^{2-N}}{{}^1\!/\!_2\,(N-1)(N-2)(B-A)^2}\;\;. \end{aligned} \tag{3}$$

In the (improper) limit $A \to -\infty$, $B \to \infty$:

$$p(\mathbf{x}) \propto \frac{1}{(x_{\max} - x_{\min})^{N-2}}\;\;. \tag{4}$$

An interpretation of this could be that we treat all $x_{\max}$ and $x_{\min}$ equally and then predict the rest of the values uniformly between them. For moderate-sized datasets using this limit won't differ much from any broad proper prior, although the marginal likelihood is now zero, so cannot be used for model comparison.

Another common special case is that we know $[A, B] = [0, 1]$, then:

$$p(\mathbf{x}) = \frac{(x_{\max} - x_{\min})^{2-N} + 1 - (1 - x_{\min})^{2-N} - (x_{\max})^{2-N}}{{}^1\!/\!_2\,(N-1)(N-2)}\;\;. \tag{5}$$

These results will be probably be found in many contexts. For a recent concrete example, both equations (4) and (5) were used in Bovy et al. (2009).

References

J. Bovy, I. Murray, and D. W. Hogg. The gravitational force law in the solar system, 2009. arXiv:0903.5308.