Contextual Lumpability

Jane Hillston

School of Informatics
The University of Edinburgh
Scotland

joint work with Carla Piazza (Udine),
Andrea Marin and Sabina Rossi (Ca' Foscari),

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- In the context of stochastic process algebras, Markovian bisimulations, have been shown to characterise lumpability in the sense that partitioning the state space on the basis of such an equivalence relation generates a lumpable partition of the underlying state space.
- In the stochastic process algebra PEPA, this equivalence relation is termed strong equivalence.

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- We also define a new equivalence relation for PEPA models, termed lumpable bisimilarity.
- We show that lumpable bisimilarity is a characterisation of contextual lumpability and induces the largest contextual lumping of the underlying Markov chain.
- Finally we provide an algorithm for lumpable bisimilarity.

Outline

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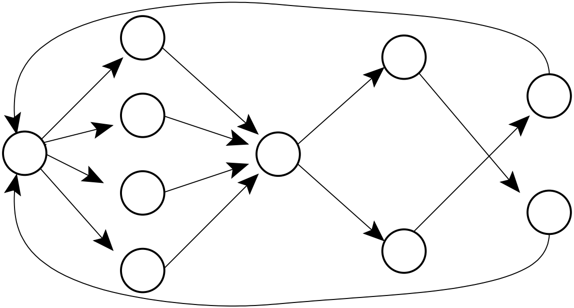
Models based on model aggregation/state-state equivalence will not in general be exact, because the macro states will not in general preserve the Markov property.

But it has long been established that if the aggregation is based on a lumpable partition then the Markov property is preserved [Kemeny and Snell, 1960].

Reducing by lumpability



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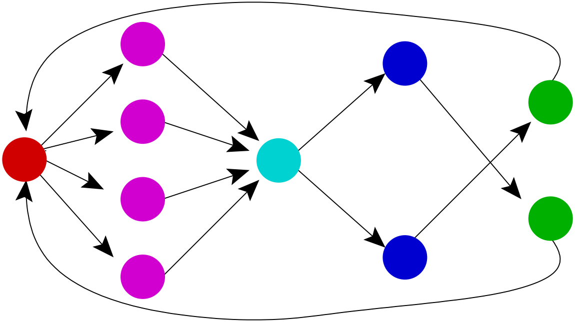

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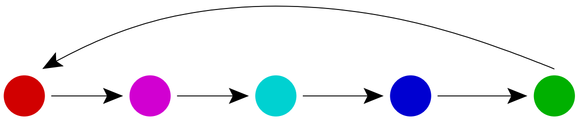

Lumpability

Lumpability

A CTMC, $\{X_i\}$, is (ordinarily) lumpable with respect to some partition $\chi = \{X_{[k]}\}$ if for any $X_{[k]}, X_{[l]} \in \chi$ with $k \neq l$ and $X_i, X_j \in X_{[k]}$,

$$q(X_i, X_{[l]}) = q(X_j, X_{[l]})$$

where $q(X_i, X_{[l]})$ is the aggregated transition rate from X_i to all states in $X_{[l]}$, i.e., $q(X_i, X_{[l]}) = \sum_{X_m \in X_{[l]}} q(X_i, X_m)$.

Lumpable Relation

Lumpable relation

A relation $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ over PEPA components is lumpable if for any component P, $ds(P)/\mathcal{R}$ induces a lumpable partition on the state space of the CTMC corresponding to P.

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Proposition

Let I be a set of indices and $\mathcal{R}_i$ be a lumpable relation for all $i \in I$. Then the union, $\mathcal{R} = \cup_{i \in I} \mathcal{R}_i$, is also a lumpable relation.

Stochastic Process Algebra

- Models are constructed from components which engage in activities.

$$(\alpha, r).P$$

action type or name

activity rate (parameter of an exponential distribution)

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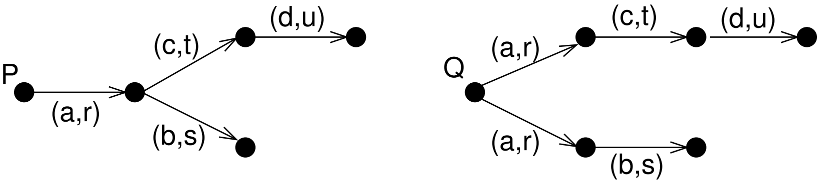
HIDING:	P/L	abstraction $\alpha \in L \Rightarrow \alpha \rightarrow \tau$

Equivalence Relations

Process algebras often define equivalence relations in terms of bisimulations, based on the notion of observability.



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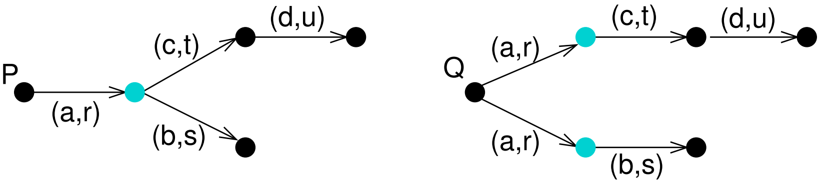

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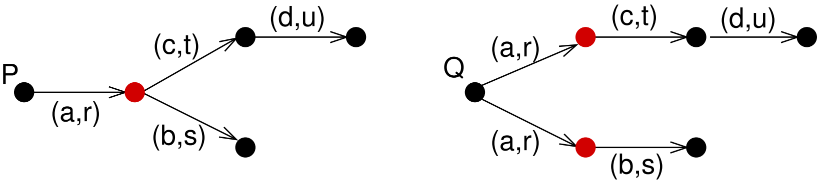

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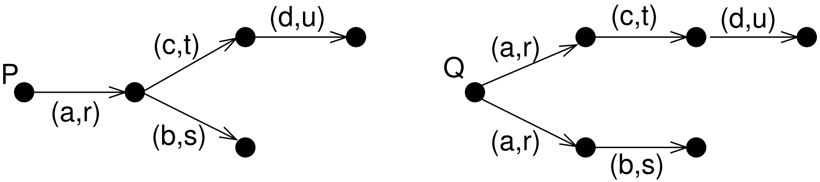

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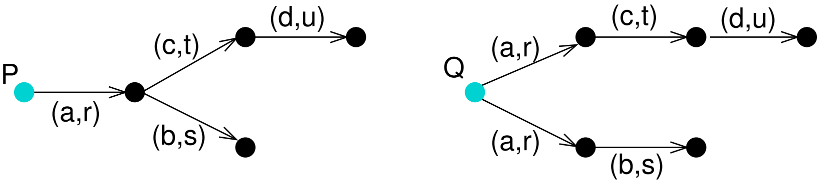

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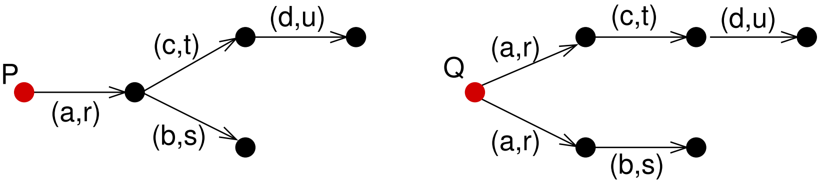

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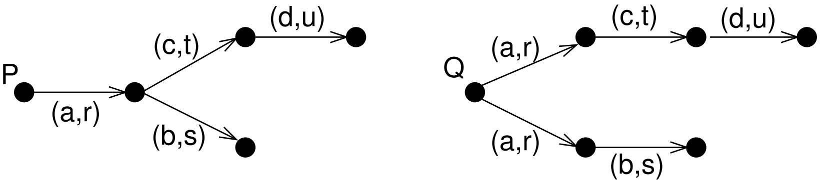

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Strong Equivalence in PEPA

Strong Equivalence

An equivalence relation $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ is a strong equivalence if whenever $(P, Q) \in \mathcal{R}$ then for all $\alpha \in \mathcal{A}$ and for all $S \in \mathcal{C}/\mathcal{R}$

$$q[P, S, \alpha] = q[Q, S, \alpha].$$

where

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Current action type preservation

A relation $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ over PEPA components is current action type preserving if for all PEPA components P, Q such that $(P, Q) \in \mathcal{R}$, $\mathcal{A}(P) = \mathcal{A}(Q)$.

Contextuality

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A relation $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$ over PEPA components is contextual if for all PEPA components P, Q such that $(P, Q) \in \mathcal{R}$ and for all contexts $C[\cdot]$,

$$(C[P], C[Q]) \in \mathcal{R}$$

where a context is a term with a hole $[\cdot]$ defined by the grammar:

$$C[\cdot] ::= [\cdot] \;\mid\; [\cdot] \underset{L}{\bowtie} P \;\mid\; P \underset{L}{\bowtie} [\cdot] \;\mid\; [\cdot]/L$$

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- Given this definition it is fairly straightforward to show that if we consider strong equivalence of states within a single model, it induces an ordinarily lumpable partition on the state space of the underlying Markov chain.
- Moreover it can be shown that strong equivalence is contextual. (Indeed, it is a congruence with respect to all operators of PEPA including the dynamic ones.)
- This means that aggregation based on lumpability can be applied component by component.

Producer-Consumer Example

- We consider a system consisting of a Producer process and a Consumer process.
- The Producer thinks, computes and transfers results to the Consumer.
- Periodically errors occur during the computation and the process has to undergo recovery before it is ready to redo the computation.
- The Consumer enqueues the transferred jobs and sends them in batches.
- Jobs waiting in the queue can spontaneously generate further jobs.
- A maximum buffer size of N jobs is set and, in case of saturation, further arrivals are lost.

Producer-Consumer Example: PEPA

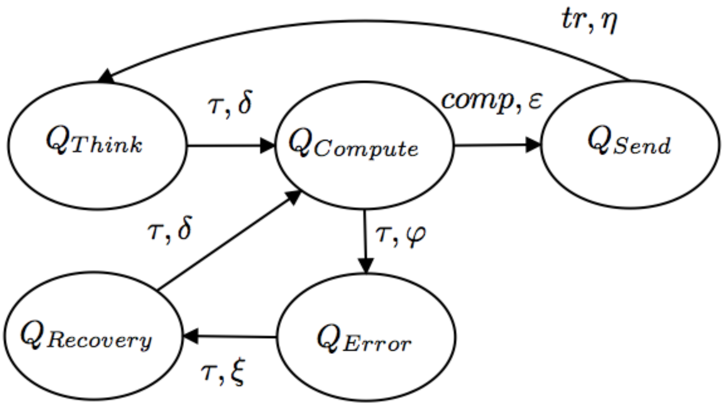
Producer

$$\begin{aligned}
Q_{Think} &\stackrel{def}{=} (\tau, \delta).Q_{Compute} \\
Q_{Compute} &\stackrel{def}{=} (comp, \varepsilon).Q_{Send} + (\tau, \varphi).Q_{Error} \\
Q_{Send} &\stackrel{def}{=} (tr, \eta).Q_{Think} \\
Q_{Error} &\stackrel{def}{=} (\tau, \xi).Q_{Recovery} \\
Q_{Recovery} &\stackrel{def}{=} (\tau, \delta).Q_{Compute}
\end{aligned}$$

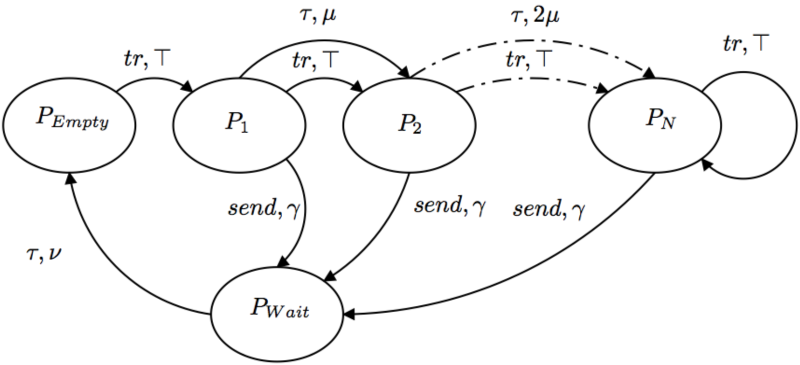
Consumer

$$\begin{aligned}
P_{Empty} &\stackrel{def}{=} (tr, \top).P_1 \\
P_i &\stackrel{def}{=} (\tau, i\mu).P_{i+1} + (tr, \top).P_{i+1} + (send, \gamma).P_{Wait} \quad 1 \leq i < N \\
P_N &\stackrel{def}{=} (tr, \top).P_N + (send, \gamma).P_{Wait} \\
P_{Wait} &\stackrel{def}{=} (\tau, \nu).P_{Empty}
\end{aligned}$$

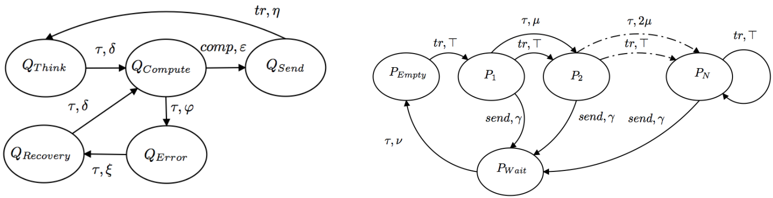
Producer-Consumer Example: Producer



Producer-Consumer Example: Consumer



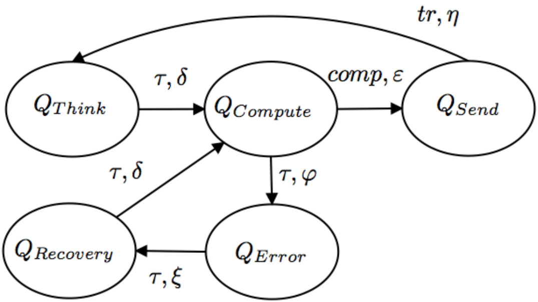
Producer-Consumer Example: System



The whole system is modelled by:

$$S \stackrel{def}{=} P_{Empty} \underset{\{tr\}}{\bowtie} Q_{Think} \,.$$

Example revisited: Original producer



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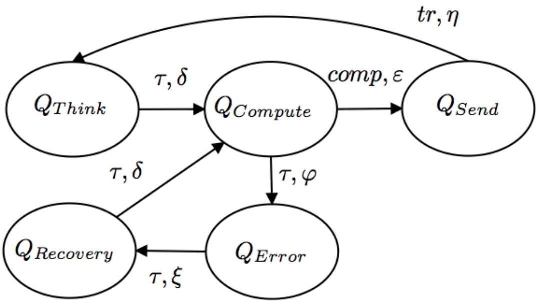

We can see that Q_{Think} and $Q_{Recovery}$ are strongly equivalent and consequently reduce the representation of the producer.

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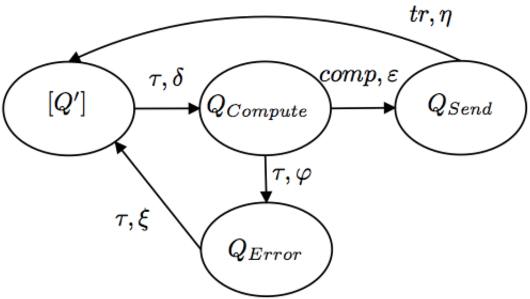

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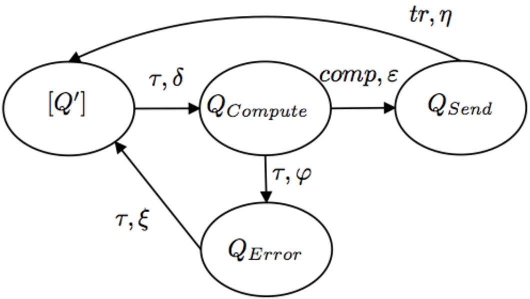

We can see that Q_{Think} and $Q_{Recovery}$ are strongly equivalent and consequently reduce the representation of the producer.

No such reduction is possible for the Consumer.

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On the other hand, the contextually of the relation — the fact that it is preserved by the static combinators of the language — has considerable advantages when working on model reduction.

Therefore we seek to define a relation which maintains contextually but achieves greater lumping.

Contextual Lumpability: formal definition

Contextual lumpability

Contextual lumpability, denoted $\cong_l$, is the largest contextual, current action type preserving, lumpable relation over PEPA terms.

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Thus we seek to define a bisimulation that can be seen to fit the template of contextual lumpability.

Lumpable Bisimulation: formal definition

Lumpable bisimulation

An equivalence relation over PEPA components, $\mathcal{R} \subseteq \mathcal{C} \times \mathcal{C}$, is a lumpable bisimulation if whenever $(P, Q) \in \mathcal{R}$ then for all $\alpha \in \mathcal{A}$ and for all $S \in \mathcal{C}/\mathcal{R}$ such that

- either $\alpha \neq \tau$,
- or $\alpha = \tau$ and $P, Q \notin S$,

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- Thus it can be regarded as coarser than strong equivalence.
- However, it has weaker congruence properties because although it is contextual, unlike strong equivalence it is not preserved by choice.
- Nevertheless for practical purposes this is sufficient for our needs.

Lumpable Bisimilarity

Proposition

Let I be a set of indices and $\mathcal{R}_i$ be a lumpable bisimulation for all $i \in I$. Then the transitive closure of their union, $\mathcal{R} = (\cup_{i \in I} \mathcal{R}_i)^*$, is also a lumpable bisimulation.

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Lumpable bisimilarity

Two PEPA components P and Q are lumpably bisimilar, written $P \approx_l Q$, if $(P, Q) \in \mathcal{R}$ for some lumpable bisimulation $\mathcal{R}$, i.e.,

$$\approx_l = \bigcup \{\mathcal{R} \mid \mathcal{R} \text{ is a lumpable bisimulation}\}.$$

$\approx_l$ is called lumpable bisimilarity and it is the largest symmetric lumpable bisimulation over PEPA components.

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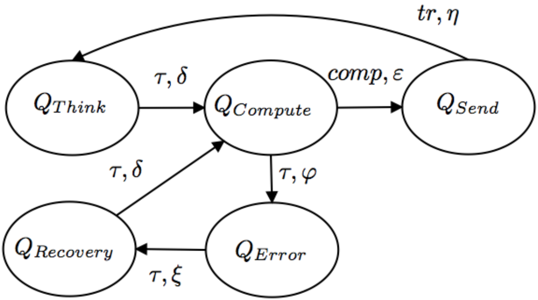
Theorem

Let P and Q be two PEPA components.
It holds that: $P \approx_l Q$ if and only if $P \cong_l Q$.

Example revisited: Original producer



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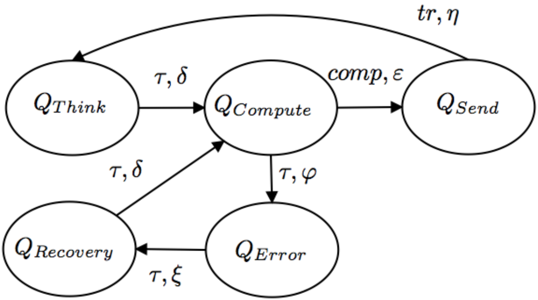

We can see that $Q_{Think} \approx_l Q_{Recovery}$ and hence they belong to the same equivalence class, say $[Q']$.

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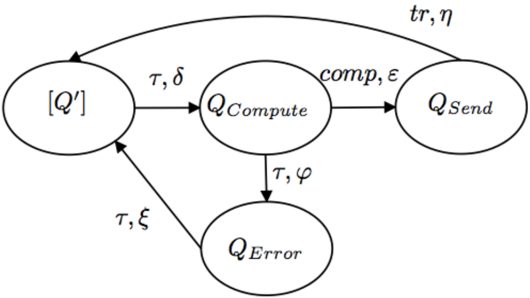

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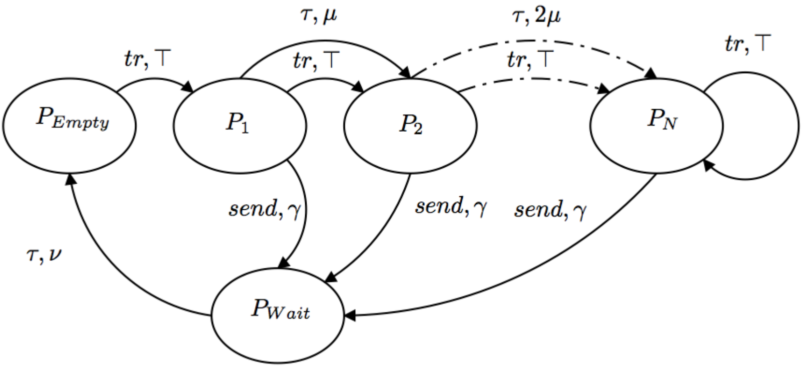
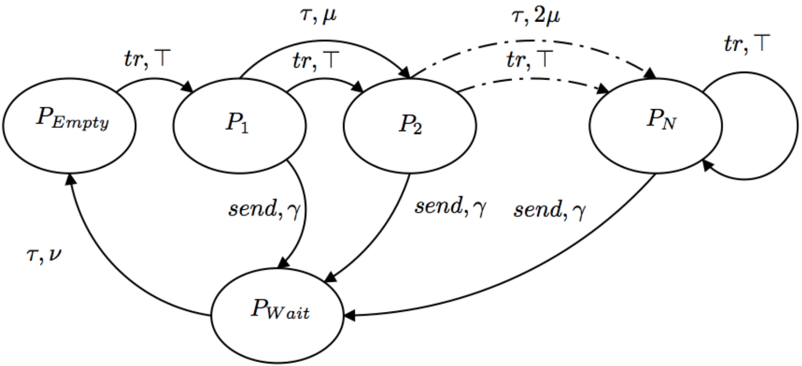
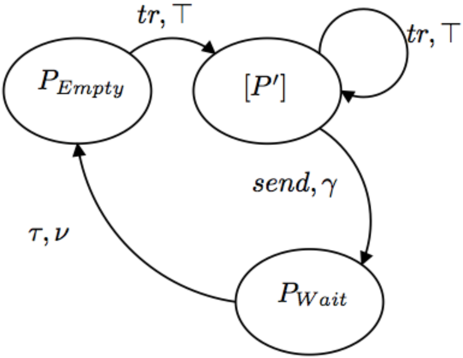


Example revisited: Original consumer



Here we observe that $P_1 \approx_l P_2 \approx_l \cdots \approx_l P_N$ and let $[P']$ be the associated equivalence class.

Example revisited: Reduced consumer



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Algorithm

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- $\textsc{Lumpability}(G = (D, \Delta, W), I)$ computes the largest equivalence relation $R \subseteq I$ over the derivative set D compatible with the derivation graph G.
- $\textsc{Contextual\_Lumpability}(\{G_\alpha^{\mathcal{D}}\}_{\alpha \in \mathcal{A} \setminus \{\tau\}}, \widehat{G}_\tau^{\mathcal{D}})$ returns the relation $\approx_I \cap (\mathcal{D} \times \mathcal{D})$, i.e., the largest lumpable bisimulation over $\mathcal{D}$.

Algorithm

$\textsc{Contextual\_Lumpability}(\{G_\alpha^{\mathcal{D}}\}_{\alpha \in \mathcal{A} \setminus \{\tau\}}, \widehat{G}_\tau^{\mathcal{D}})$

Require: $\mathcal{D}$ finite and $\mathcal{D} = ds(\mathcal{D})$;
$R = \emptyset$;
$\mathcal{R} = \mathcal{D} \times \mathcal{D}$;
while $R \neq \mathcal{R}$ **do**
  $R = \mathcal{R}$;
  for all $\alpha \in \mathcal{A} \setminus \{\tau\}$ **do**
    $\mathcal{R} = \textsc{Lumpability}(G_\alpha^{\mathcal{D}}, \mathcal{R})$;
  end for
  $\mathcal{R} = \textsc{Lumpability}(\widehat{G}_\tau^{\mathcal{D}}, \mathcal{R})$;
end while
return $\mathcal{R}$;

Conclusions

- We have introduced a relaxed form of lumpability for PEPA models, termed contextual lumpability.
- We also defined a new equivalence relation for PEPA models, termed lumpable bisimilarity.
- We showed that lumpable bisimilarity is a characterisation of contextual lumpability and induces the largest contextual lumping of the underlying Markov chain of a PEPA model.
- Finally we provided an algorithm for lumpable bisimilarity.