Fluid Approximation for the Analysis of Collective Systems

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Outline

1. Introduction
   - Collective Systems
   - Quantitative Analysis
   - Stochastic Process Algebra
2. Quantitative Analysis of Collective Systems
   - Model construction
   - Mathematical analysis: fluid approximation
   - Numerical illustration
   - Deriving properties: fluid model checking
3. Conclusions

Outline

Collective Systems

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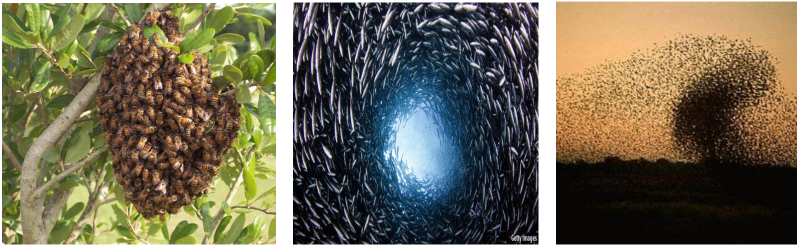
in the natural world ....



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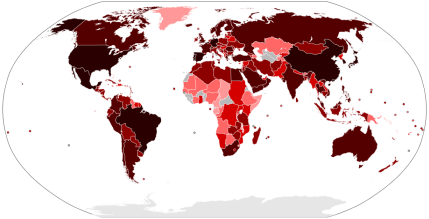

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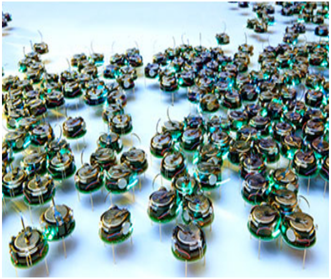


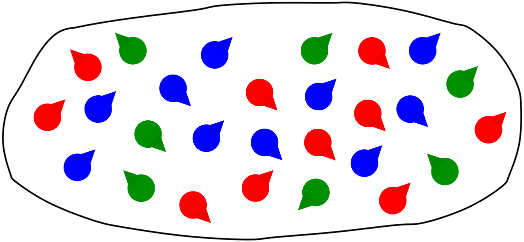
Collective Systems

We are surrounded by examples of collective systems:

an informatic environment

Collective Systems

From a computer science perspective these systems can be viewed as being made up of a large number of interacting entities.

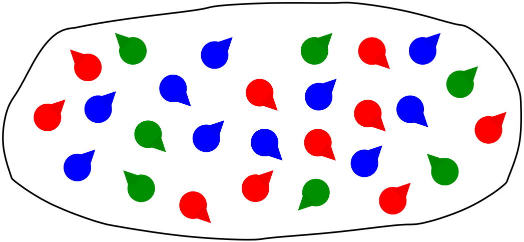


Each entity may have its own properties, objectives and actions.

At the system level these combine to create the collective behaviour.

Collective Systems

The behaviour of the system is thus dependent on the behaviour of the individual entities.



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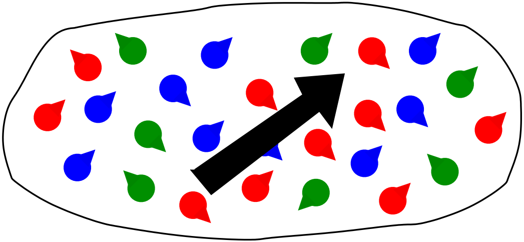

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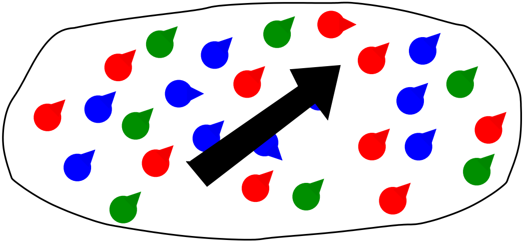

And the behaviour of the individuals will be influenced by the state of the overall system.

The Informatic Environment

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For instance, may examples of such systems can be found in components of Smart Cities, such as smart urban transport and smart grid electricity generation and storage.

Quantitative Modelling

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Originally queueing networks were primarily used to construct models, and sophisticated analysis techniques were developed.

These techniques are no longer widely applicable for expressing the dynamic behaviour observed in distributed systems, and this is even more true of systems with collective behaviour.

Performance Modelling: Motivation

Capacity Planning

- How many clients can the existing server support and maintain reasonable response times?
- How many buses do I need to maintain service at peak time in a smart urban transport system?

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System Configuration

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- What capacity do I need at bike stations to minimise the movement of bikes by truck?

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System Tuning

- In an automated factory what speed of conveyor belt will minimize robot idle time and jamming but maximize throughput?
- What strategy can I use to maintain supply-demand balance within a smart electricity grid?

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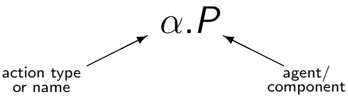
Primary examples include:

- Stochastic Petri Nets and
- Stochastic/Markovian Process Algebras.

Process Algebra

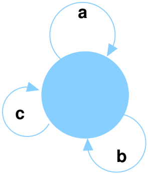
- Models consist of agents which engage in actions.



$\alpha.P$

action type or name

agent/ component



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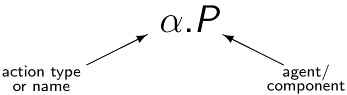

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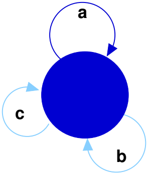

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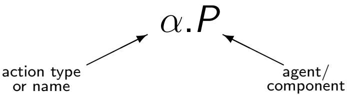

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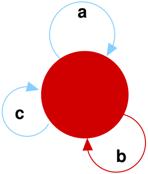

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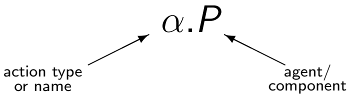

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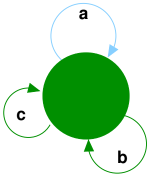

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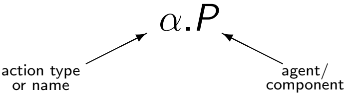

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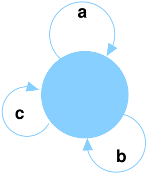

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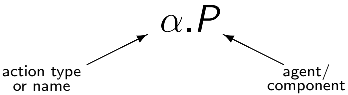

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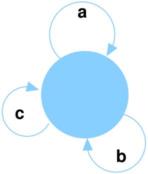

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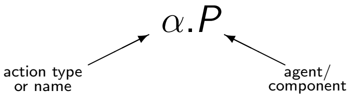

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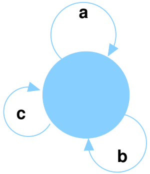

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Process algebra
model

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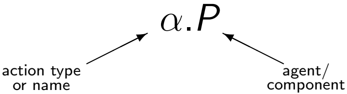

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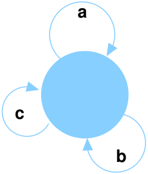

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Process algebra model $\xrightarrow{\text{SOS rules}}$

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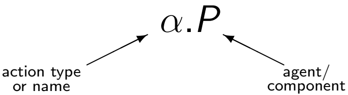

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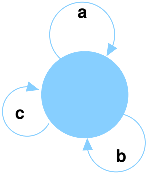

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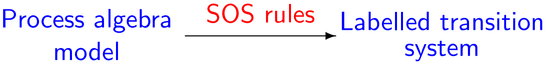

Process algebra model $\xrightarrow{\text{SOS rules}}$ Labelled transition system

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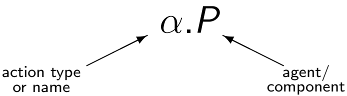
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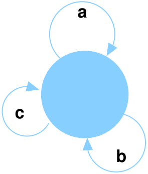

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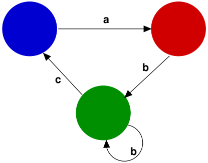

Stochastic process algebras

Process algebras where models are decorated with quantitative information used to generate a stochastic process are stochastic process algebras (SPA).

Stochastic Process Algebra

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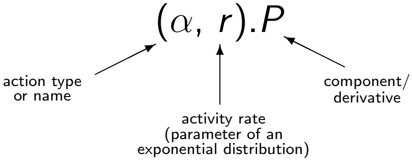

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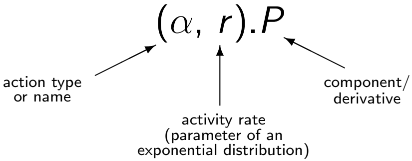

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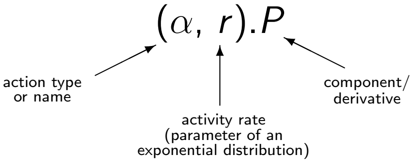

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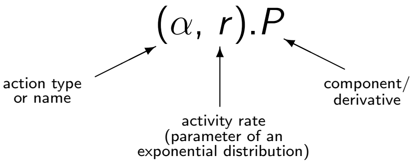

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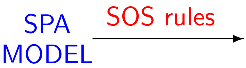

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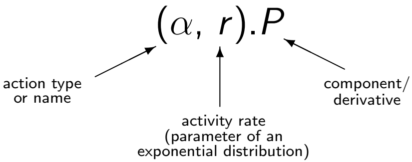

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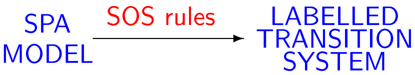

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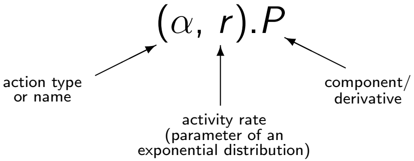

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SOS rules

LABELLED
TRANSITION
SYSTEM

state transition

diagram

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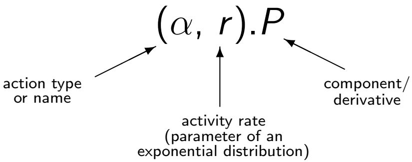

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SPA MODEL $\xrightarrow{\text{SOS rules}}$ LABELLED TRANSITION SYSTEM $\xrightarrow[\text{diagram}]{\text{state transition}}$ CTMC **Q**

Integrated analysis

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Reachability analysis

How long will it take
for the system to arrive
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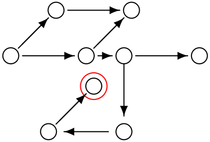

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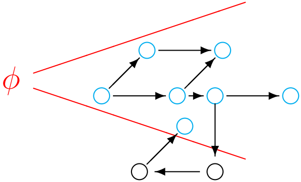
Model checking

Does a given property ϕ hold within the system with a given probability?

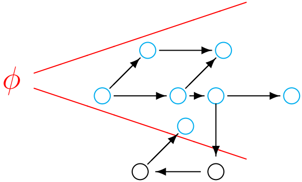


Integrated analysis

Qualitative verification can now be complemented by quantitative verification.

Model checking

For a given starting state
how long is it until
a given property ϕ holds?



Performance Evaluation Process Algebra

$(\alpha, f).P$	Prefix
$P_1 + P_2$	Choice
$P_1 \underset{L}{\bowtie} P_2$	Co-operation
P/L	Hiding
X	Constant

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When working with large numbers of entities, we write $P[n]$ to denote an array of n copies of P executing in parallel.

$$P[5] \equiv (P \parallel P \parallel P \parallel P \parallel P)$$

Structured Operational Semantics

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Choice

$$\frac{E \xrightarrow{(\alpha,r)} E'}{E + F \xrightarrow{(\alpha,r)} E'}$$

$$\frac{F \xrightarrow{(\alpha,r)} F'}{E + F \xrightarrow{(\alpha,r)} F'}$$

Structured Operational Semantics: Cooperation ($\alpha \notin L$)

Cooperation

$$\frac{E \xrightarrow{(\alpha,r)} E'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,r)} E' \underset{L}{\bowtie} F} \, (\alpha \notin L)$$

Structured Operational Semantics: Cooperation ($\alpha \notin L$)

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Structured Operational Semantics: Cooperation ($\alpha \in L$)

Cooperation $$\frac{E \xrightarrow{(\alpha,r_1)} E' \qquad F \xrightarrow{(\alpha,r_2)} F'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,R)} E' \underset{L}{\bowtie} F'} \; (\alpha \in L)$$

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$$\text{where} \quad R = \frac{r_1}{r_\alpha(E)} \frac{r_2}{r_\alpha(F)} min(r_\alpha(E), r_\alpha(F))$$

Apparent Rate

$$
\begin{aligned}
r_\alpha((\beta, r).P) &= \begin{cases} r & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases} \\
r_\alpha(P + Q) &= r_\alpha(P) + r_\alpha(Q) \\
r_\alpha(A) &= r_\alpha(P) \qquad \text{where } A \overset{def}{=} P \\
r_\alpha(P \underset{L}{\bowtie} Q) &= \begin{cases} r_\alpha(P) + r_\alpha(Q) & \alpha \notin L \\ min(r_\alpha(P), r_\alpha(Q)) & \alpha \in L \end{cases} \\
r_\alpha(P/L) &= \begin{cases} r_\alpha(P) & \alpha \notin L \\ 0 & \alpha \in L \end{cases}
\end{aligned}
$$

Structured Operational Semantics: Hiding

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$$\frac{E \xrightarrow{(\alpha,r)} E'}{E/L \xrightarrow{(\alpha,r)} E'/L} \, (\alpha \notin L)$$

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Structured Operational Semantics: Constants

Constant

$$\frac{E \xrightarrow{(\alpha,r)} E'}{A \xrightarrow{(\alpha,r)} E'} \, (A \overset{\mathit{def}}{=} E)$$

A simple example: processors and resources

$$
\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\overset{def}{=} (task1, r_3).Res_1 \\
Res_1 &\overset{def}{=} (reset, r_4).Res_0
\end{aligned}
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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

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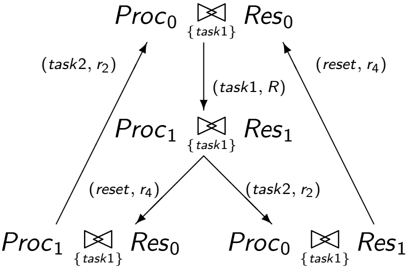

$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$

$(task1, R)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_1$

$(reset, r_4)$

$(task2, r_2)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_0$

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$$R = \min(r_1, r_3)$$

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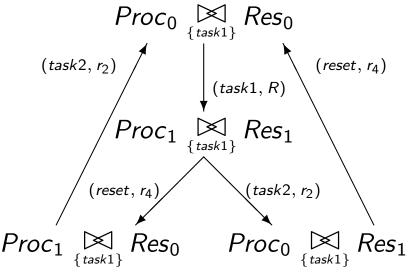


$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$

$(task2, r_2)$

$(task1, R)$

$(reset, r_4)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_1$

$(reset, r_4)$

$(task2, r_2)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_0$

$Proc_0 \underset{\{task1\}}{\bowtie} Res_1$

$$R = \min(r_1, r_3)$$

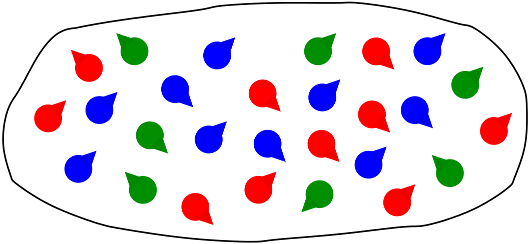
$$
\mathbf{Q} = \begin{pmatrix}
-R & R & 0 & 0 \\
0 & -(r_2 + r_4) & r_4 & r_2 \\
r_2 & 0 & -r_2 & 0 \\
r_4 & 0 & 0 & -r_4
\end{pmatrix}
$$

Outline

1. Introduction
   - Collective Systems
   - Quantitative Analysis
   - Stochastic Process Algebra
2. Quantitative Analysis of Collective Systems
   - Model construction
   - Mathematical analysis: fluid approximation
   - Numerical illustration
   - Deriving properties: fluid model checking
3. Conclusions

Modelling collective behaviour

A key feature of collective systems is the existence of populations of entities who share certain characteristics.



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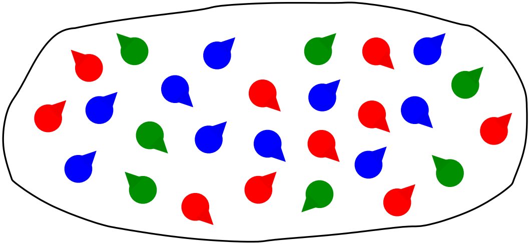

High-level modelling formalisms allow this repetition to be captured at the high-level rather than explicitly.

Process Algebra for Collective Systems

Process algebra are well-suited for constructing models of collective systems:

- Developed to represent concurrent behaviour compositionally;

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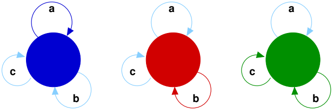
Recent advances in analysis techniques for process algebras have made it possible to study such systems even when the number of entities and activities become huge.

Solving discrete state models



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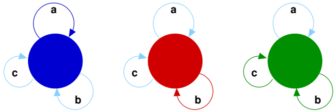

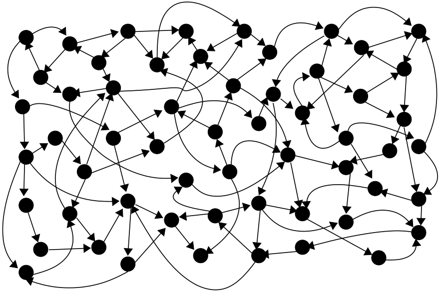

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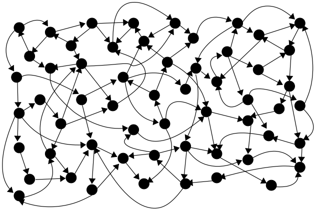

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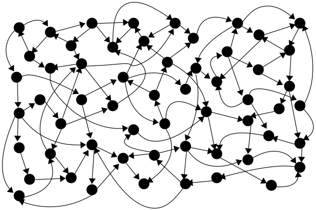

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$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,N} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,N} \\ \vdots & \vdots & & \vdots \\ q_{N,1} & q_{N,2} & \cdots & q_{N,N} \end{pmatrix}$$

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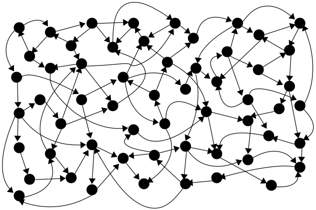

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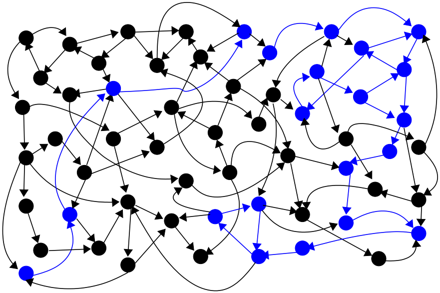
$$\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_N(t))$$

$$\pi(\infty)Q = 0$$

Solving discrete state models

Alternatively they may be studied using stochastic simulation. Each run generates a single trajectory through the state space. Many runs are needed in order to obtain average behaviours.



State space explosion

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

Identity and Individuality

Collective systems are constructed from many instances of a set of components.

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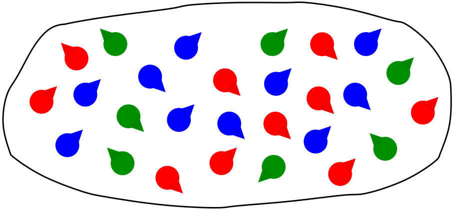

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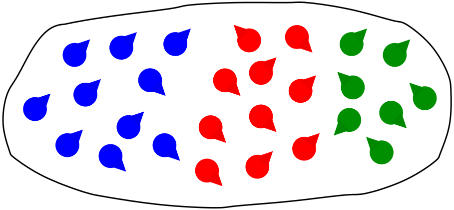

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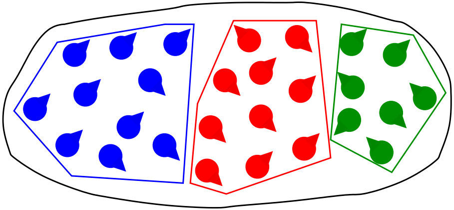

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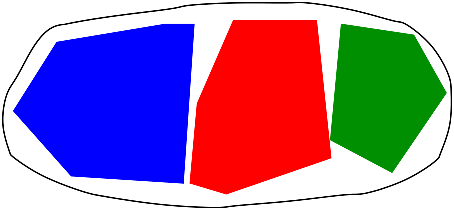

We may choose to disregard the identity of components.

Even better reductions can be achieved when we no longer regard the components as individuals.

Population statistics: emergent behaviour

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Population statistics: emergent behaviour

A shift in perspective allows us to model the interactions between individual components but then only consider the system as a whole as an interaction of populations.

This allows us to model much larger systems than previously possible but in making the shift we are no longer able to collect any information about individuals in the system.

To characterise the behaviour of a population we calculate the proportion of individuals within the population that are exhibiting certain behaviours rather than tracking individuals directly.

Furthermore we make a continuous approximation of how the proportions vary over time.

Continuous Approximation

This means shifting to a different mathematical representation, where we no longer keep track of the individual states of each entity.

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As we are focussed instead of proportions within populations we now treat these variables as continuous rather than discrete.

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Use ordinary differential equations to represent the evolution of those variables over time.

Simple example revisited

$$
\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\overset{def}{=} (task1, r_3).Res_1 \\
Res_1 &\overset{def}{=} (reset, r_4).Res_0
\end{aligned}
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$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

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\end{aligned}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

CTMC interpretation

Processors (N_P)	Resources (N_R)	States ($2^{N_P+N_R}$)
1	1	4
2	1	8
2	2	16
3	2	32
3	3	64
4	3	128
4	4	256
5	4	512
5	5	1024
6	5	2048
6	6	4096
7	6	8192
7	7	16384
8	7	32768
8	8	65536
9	8	131072
9	9	262144
10	9	524288
10	10	1048576

Simple example revisited

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\end{aligned}$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

- $task1$ decreases $Proc_0$ and Res_0
- $task1$ increases $Proc_1$ and Res_1
- $task2$ decreases $Proc_1$
- $task2$ increases $Proc_0$
- $reset$ decreases Res_1
- $reset$ increases Res_0

Simple example revisited

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\end{aligned}$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -\min(r_1\, x_1, r_3\, x_3) + r_2\, x_2$$

$$x_1 = \text{no. of } Proc_1$$

- $task1$ decreases $Proc_0$
- $task1$ is performed by $Proc_0$ and Res_0
- $task2$ increases $Proc_0$
- $task2$ is performed by $Proc_1$

Simple example revisited

$$
\begin{array}{rcl}
Proc_0 & \stackrel{def}{=} & (task1, r_1).Proc_1 \\
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Res_1 & \stackrel{def}{=} & (reset, r_4).Res_0
\end{array}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

ODE interpretation

$$\frac{dx_1}{dt} = -\min(r_1\, x_1, r_3\, x_3) + r_2\, x_2$$

$$x_1 = \text{no. of } Proc_1$$

$$\frac{dx_2}{dt} = \min(r_1\, x_1, r_3\, x_3) - r_2\, x_2$$

$$x_2 = \text{no. of } Proc_2$$

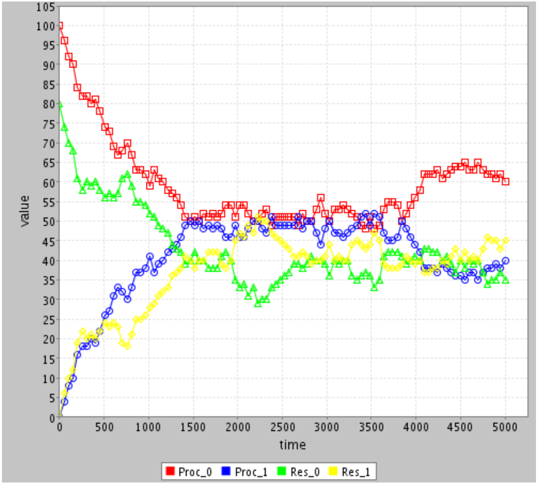
$$\frac{dx_3}{dt} = -\min(r_1\, x_1, r_3\, x_3) + r_4\, x_4$$

$$x_3 = \text{no. of } Res_0$$

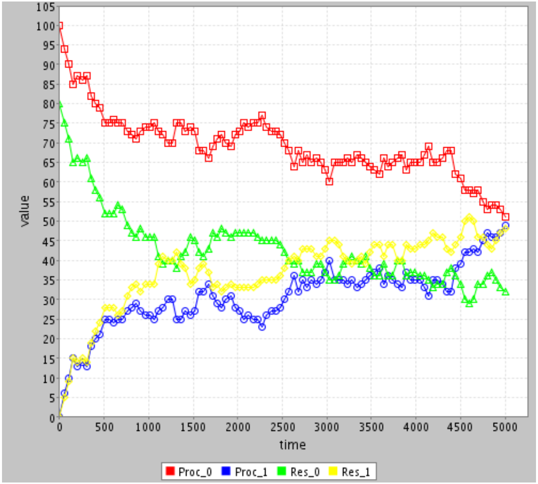
$$\frac{dx_4}{dt} = \min(r_1\, x_1, r_3\, x_3) - r_4\, x_4$$

$$x_4 = \text{no. of } Res_1$$

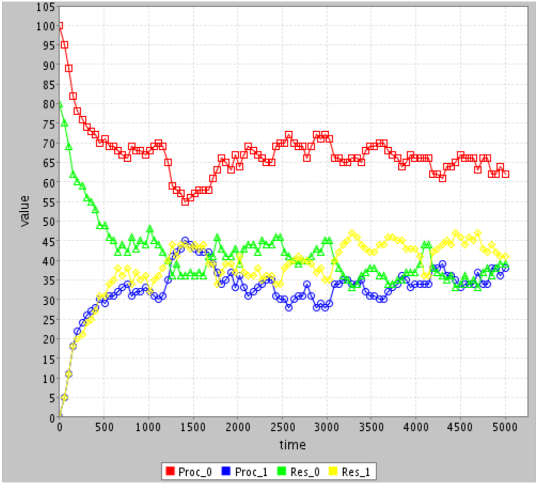
100 processors and 80 resources (simulation run A)



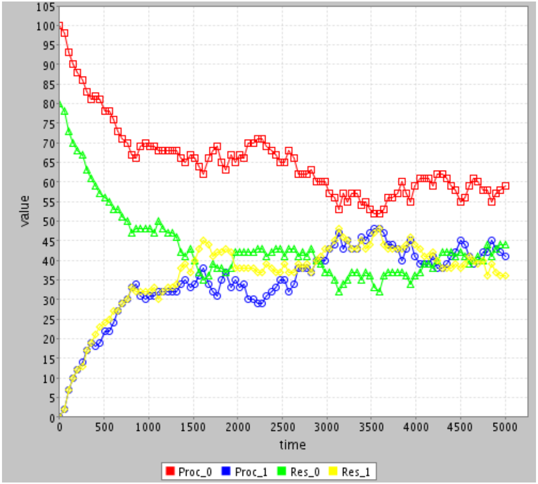
100 processors and 80 resources (simulation run B)



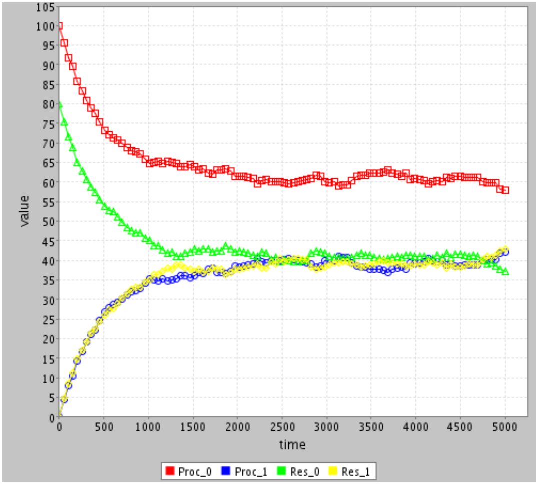
100 processors and 80 resources (simulation run C)



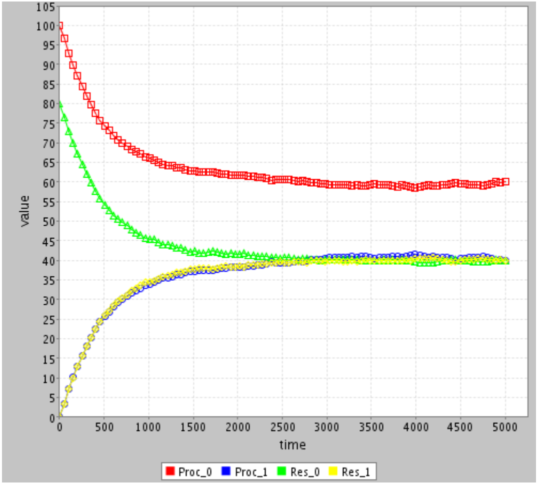
100 processors and 80 resources (simulation run D)



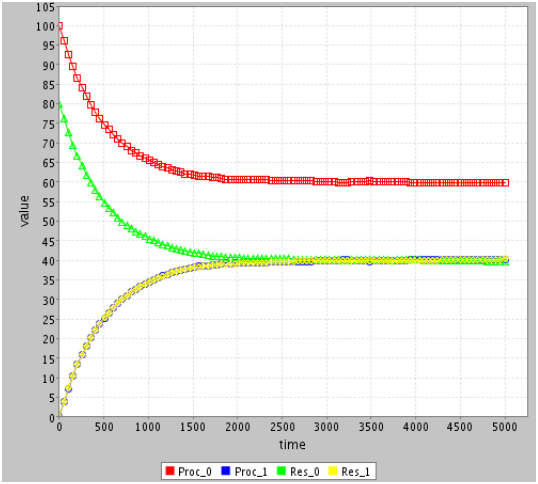
100 processors and 80 resources (average of 10 runs)



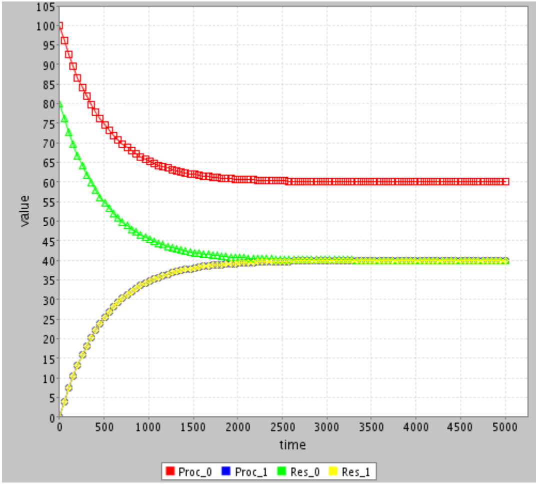
100 Processors and 80 resources (average of 100 runs)



100 processors and 80 resources (average of 1000 runs)



100 processors and 80 resources (ODE solution)



Deriving a Fluid Approximation of a SPA model

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SPA MODEL

SOS rules

LABELLED TRANSITION SYSTEM

state transition diagram

CTMC **Q**

Deriving a Fluid Approximation of a SPA model

The aim is to represent the CTMC implicitly (avoiding state space explosion), and to generate the set of ODEs which are the fluid limit of that CTMC.

The existing (CTMC) SOS semantics is not suitable for this purpose because it constructs the state space of the CTMC explicitly.

We define a structured operational semantics which defines the possible transitions of an arbitrary abstract state and from this derive the ODEs.

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We define a structured operational semantics which defines the possible transitions of an arbitrary abstract state and from this derive the ODEs.

SPA MODEL

SOS rules

SYMBOLIC LABELLED TRANSITION SYSTEM

generator functions

ABSTRACT CTMC **Q** or ODEs $F_{\mathcal{M}}(x)$

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3. Calculate the rate of the transitions in terms of an arbitrary state of the CTMC.

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2. Collect the transitions of the reduced context as symbolic updates on the state representation (Jump Multiset)
3. Calculate the rate of the transitions in terms of an arbitrary state of the CTMC.

Once this is done we can extract the vector field $F_{\mathcal{M}}(x)$ from the jump multiset, under the assumption that the population size tends to infinity.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

Context Reduction

$$
\begin{array}{rcl}
Proc_0 & \overset{def}{=} & (task1, r_1).Proc_1 \\
Proc_1 & \overset{def}{=} & (task2, r_2).Proc_0 \\
Res_0 & \overset{def}{=} & (task1, r_3).Res_1 \\
Res_1 & \overset{def}{=} & (reset, r_4).Res_0 \\
System & \overset{def}{=} & Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{array}
$$

$$\Downarrow$$

$$\mathcal{R}(System) = \{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\}$$

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Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

Location Dependency

$$System \overset{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

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$$\{\mathit{Proc}_0, \mathit{Proc}_1\} \underset{\{\mathit{task1}\}}{\bowtie} \{\mathit{Res}_0, \mathit{Res}_1\} \parallel \{\mathit{Proc}_0, \mathit{Proc}_1\}$$

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Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
\mathit{Proc_0} &\overset{\mathit{def}}{=} (\mathit{task1}, r_1).\mathit{Proc_1} \\
\mathit{Proc_1} &\overset{\mathit{def}}{=} (\mathit{task2}, r_2).\mathit{Proc_0} \\
\mathit{Res_0} &\overset{\mathit{def}}{=} (\mathit{task1}, r_3).\mathit{Res_1} \\
\mathit{Res_1} &\overset{\mathit{def}}{=} (\mathit{reset}, r_4).\mathit{Res_0} \\
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\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\frac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_{*} Proc_1}$$

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Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
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$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\frac{\dfrac{Proc_0 \xrightarrow{task1,r_1} Proc_1}{Proc_0 \xrightarrow{task1,r_1\xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1,r_3} Res_1}{Res_0 \xrightarrow{task1,r_3\xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

Apparent Rate Calculation

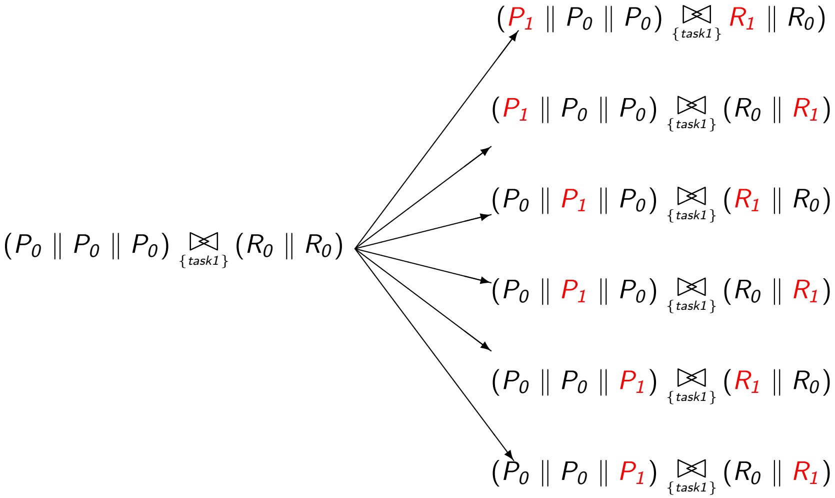
$$\dfrac{\dfrac{\mathit{Proc0} \xrightarrow{\mathit{task1}, r_1} \mathit{Proc1}}{\mathit{Proc0} \xrightarrow{\mathit{task1}, r_1\xi_1}_* \mathit{Proc1}} \qquad \dfrac{\mathit{Res0} \xrightarrow{\mathit{task1}, r_3} \mathit{Res1}}{\mathit{Res0} \xrightarrow{\mathit{task1}, r_3\xi_3}_* \mathit{Res1}}}{\mathit{Proc0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res0} \xrightarrow{\mathit{task1}, r(\xi)}_* \mathit{Proc1} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res1}}$$

Apparent Rate Calculation

$$\dfrac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

$$r(\xi) = \min\left(r_1 \xi_1, r_3 \xi_4\right)$$

$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC



$(P_1 \parallel P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} R_1 \parallel R_0)$

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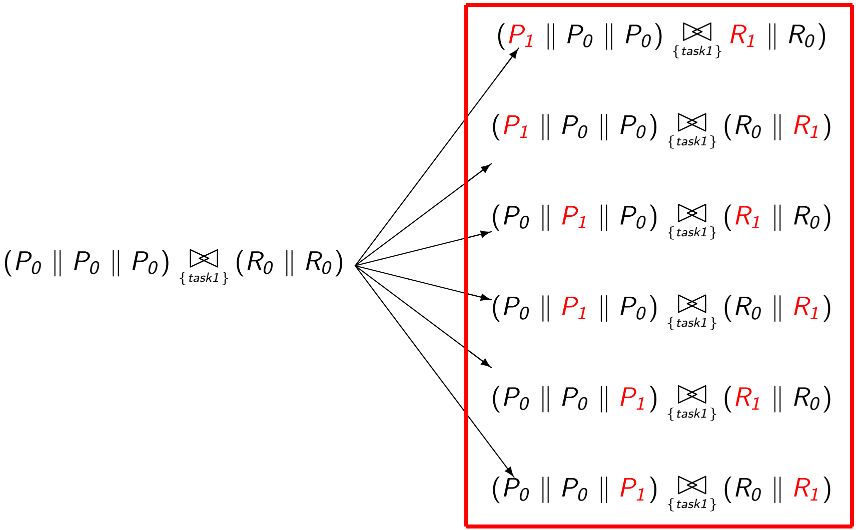

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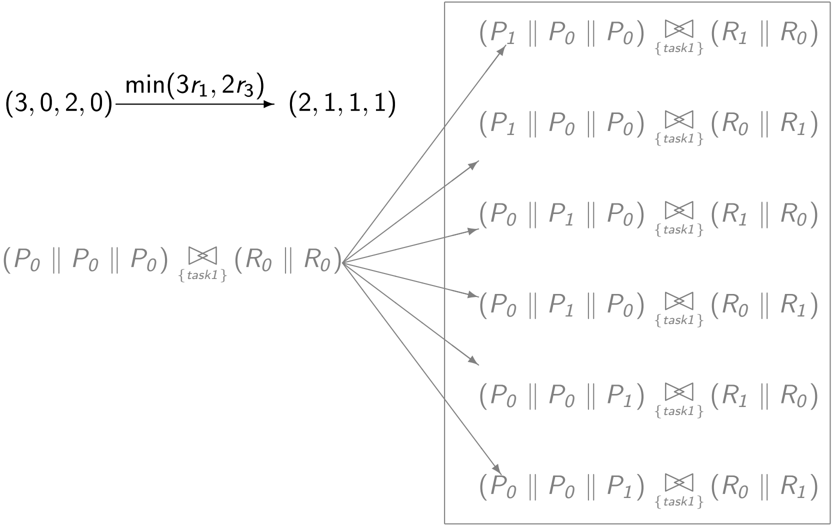
$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(3,0,2,0) \xrightarrow{\min(3r_1, 2r_3)} (2,1,1,1)$$



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Jump Multiset

$$\mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0} \xrightarrow{\mathit{task1},\, r(\xi)}_{*} \mathit{Proc_1} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_1}$$

$$r(\xi) = \min\left(r_{\mathit{1}}\xi_1, r_{\mathit{3}}\xi_3\right)$$

Jump Multiset

$$\mathit{Proc}_0 \underset{\{task1\}}{\bowtie} \mathit{Res}_0 \xrightarrow{task1,\ r(\xi)}_{*} \mathit{Proc}_1 \underset{\{task1\}}{\bowtie} \mathit{Res}_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$\mathit{Proc}_1 \underset{\{task1\}}{\bowtie} \mathit{Res}_0 \xrightarrow{task2,\ \xi_2 r_2}_{*} \mathit{Proc}_0 \underset{\{task1\}}{\bowtie} \mathit{Res}_0$$

Jump Multiset

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$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2,\ \xi_2 r_2}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Equivalent Transitions

Some transitions may give the same information:

$$\begin{array}{l} Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\, \xi_4 r_4}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \\ Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\, \xi_4 r_4}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \end{array}$$

i.e., Res_1 may perform an action independently from the rest of the system.

This is captured by the procedure used for the construction of the generator function $f(\xi, l, \alpha)$

Construction of $f(\xi, l, \alpha)$

$$\mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_1} \xrightarrow{\mathit{reset},\ \xi_4 r_4}_{*} \mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0}$$

Construction of $f(\xi, l, \alpha)$

$$\mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_1} \xrightarrow{\mathit{reset},\, \xi_4 r_4}{}_* \mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0}$$

- Take $l = (0, 0, 0, 0)$

Construction of $f(\xi, I, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $I = (0, 0, 0, 0)$
- Add -1 to all elements of I corresponding to the indices of the components in the lhs of the transition

$$I = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of I corresponding to the indices of the components in the rhs of the transition

$$I = (-1 + 1, 0, +1, -1) = (0, 0, +1, -1)$$

Construction of $f(\xi, I, \alpha)$

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$$f\big(\xi, (0, 0, +1, -1), reset\big) = \xi_4 r_4$$

Construction of $f(\xi, l, \alpha)$

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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

Construction of $f(\xi, l, \alpha)$

$$
\begin{array}{lcl}
Proc_0 \underset{\{task1\}}{\bowtie} Res_0 & \xrightarrow{task1,\ r(\xi)}_* & Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \\
Proc_1 \underset{\{task1\}}{\bowtie} Res_0 & \xrightarrow{task2,\ \xi_2 r_2'}_* & Proc_0 \underset{\{task1\}}{\bowtie} Res_0
\end{array}
$$

$$
\begin{array}{rcl}
f(\xi, (-1, +1, -1, +1), task1) & = & r(\xi) \\
f(\xi, (+1, -1, 0, 0), task2) & = & \xi_2 r_2
\end{array}
$$

Construction of $f(\xi, l, \alpha)$

$$\begin{aligned}
Proc_0 \underset{\{task1\}}{\bowtie} Res_0 &\xrightarrow{task1,\ r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \\
Proc_1 \underset{\{task1\}}{\bowtie} Res_0 &\xrightarrow{task2,\ \xi_2 r_2'}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \\
Proc_0 \underset{\{task1\}}{\bowtie} Res_1 &\xrightarrow{reset,\ \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0
\end{aligned}$$

$$\begin{aligned}
f(\xi, (-1, +1, -1, +1), task1) &= r(\xi) \\
f(\xi, (+1, -1, 0, 0), task2) &= \xi_2 r_2 \\
f(\xi, (0, 0, +1, -1), reset) &= \xi_4 r_4
\end{aligned}$$

Capturing behaviour in the Generator Function

$$\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
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\end{aligned}$$

Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \text{ and } \xi_3 + \xi_4 = N_R$$

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Generator Function

$$f(\xi, l, \alpha) : \quad \begin{array}{rcl}
f(\xi, (-1, 1, -1, 1), task1) & = & \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1, -1, 0, 0), task2) & = & r_2\xi_2 \\
f(\xi, (0, 0, 1, -1), reset) & = & r_4\xi_4
\end{array}$$

Extraction of the ODE from f

Generator Function

$$\begin{aligned} f(\xi,(-1,1,-1,1),\mathit{task1}) &= \min\left(r_1\xi_1, r_3\xi_3\right) \\ f(\xi,(1,-1,0,0),\mathit{task2}) &= r_2\xi_2 \\ f(\xi,(0,0,1,-1),\mathit{reset}) &= r_4\xi_4 \end{aligned}$$

Differential Equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_{\mathcal{M}}(x) = \sum_{l\in\mathbb{Z}^d} l \sum_{\alpha\in\mathcal{A}} f(x,l,\alpha)$$

$$\begin{aligned} &= (-1,1,-1,1)\min\left(r_1x_1, r_3x_3\right) + (1,-1,0,0)r_2x_2 \\ &\qquad + (0,0,1,-1)r_4x_4 \end{aligned}$$

Extraction of the ODE from f

Generator Function

$$\begin{aligned} f(\xi,(-1,1,-1,1),\mathit{task1}) &= \min\left(r_1\xi_1, r_3\xi_3\right) \\ f(\xi,(1,-1,0,0),\mathit{task2}) &= r_2\xi_2 \\ f(\xi,(0,0,1,-1),\mathit{reset}) &= r_4\xi_4 \end{aligned}$$

Differential Equation

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\min\left(r_1x_1, r_3x_3\right) + r_2x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \min\left(r_1x_1, r_3x_3\right) - r_2x_2 \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\min\left(r_1x_1, r_3x_3\right) + r_4x_4 \\ \frac{\mathrm{d}x_4}{\mathrm{d}t} &= \min\left(r_1x_1, r_3x_3\right) - r_4x_4 \end{aligned}$$

Consistency results

- The vector field $\mathcal{F}(x)$ is Lipschitz continuous i.e. all the rate functions governing transitions in the process algebra satisfy local continuity conditions.

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- We can prove this using Kurtz's theorem: Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes, T.G. Kurtz, J. Appl. Prob. (1970).

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- The generated ODEs are the fluid limit of the family of CTMCs: this family forms a sequence as the initial populations are scaled by a variable n.
- We can prove this using Kurtz's theorem: Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes, T.G. Kurtz, J. Appl. Prob. (1970).
- Moreover Lipschitz continuity of the vector field guarantees existence and uniqueness of the solution to the initial value problem.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

Quantitative properties

The derived vector field $\mathcal{F}(x)$, gives an approximation of the expected count for each population over time.

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- Fluid approximation of passage times have been defined.

R.A.Hayden, A.Stefanek and J.T.Bradley. Fluid computation of passage-time distributions in large Markov models. TCS 2012.

Fluid model checking

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Work on this is on-going but there are initial results for:

- CSL properties of a single agent within a population.

L.Bortolussi and J.Hillston. Fluid model checking. CONCUR 2012.

- The fraction of a population that satisfies a property expressed as a one-clock deterministic timed automaton.

L.Bortolussi and R.Lanciani. Central Limit Approximation for Stochastic Model Checking. QEST 2013.

CSL model checking of a single agent

We consider properties of a single agent within a population, expressed in the Continuous Stochastic Logic (CSL), usually used for model checking CTMCs.

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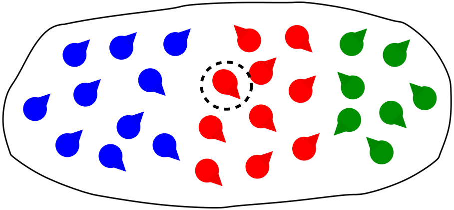
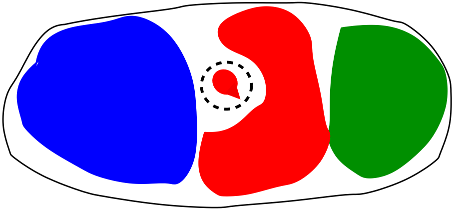
We consider an arbitrary member of the population.



CSL model checking of a single agent

We consider properties of a single agent within a population, expressed in the Continuous Stochastic Logic (CSL), usually used for model checking CTMCs.

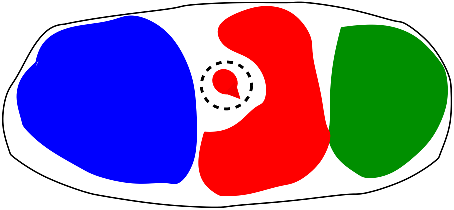
This agent is kept discrete, making transitions between its discrete states, but all other agents are treated as a mean-field influencing the behaviour of this agent.



CSL model checking of a single agent

We consider properties of a single agent within a population, expressed in the Continuous Stochastic Logic (CSL), usually used for model checking CTMCs.

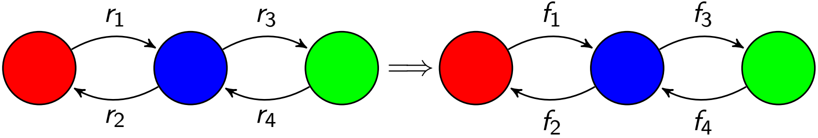
Essentially we keep a detailed discrete-event representation of the one agent and make a fluid approximation of the rest of the population.

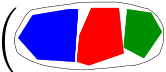


Inhomogeneous CTMC

The transition rates within the discrete-event representation will depend on the rest of the population.

i.e. it will depend on the vector field capturing the behaviour of the residual population.



where $f_i = f\Big($  $\Big)$

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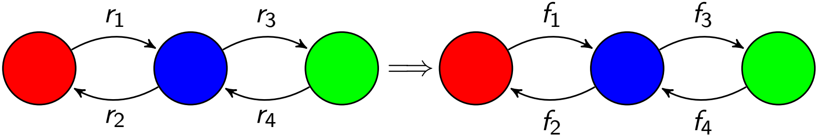
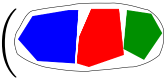


where $f_i = f\Bigg($  $\Bigg)$

It is an inhomogeneous continuous time Markov chain.

Model checking the ICTMC

Care is needed to model check the ICTMC, which proceeds by explicitly calculating the reachability probability probabilities for states of interest (analogously to CSL model checking on CTMCs).

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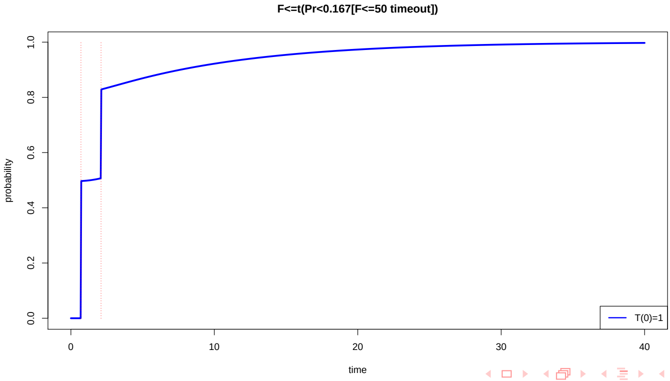
The inhomogeneous time within the model means that truth values may change with respect to time.



Outline

Conclusions

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- Their role within infrastructure, such as within smart cities, make it essential that quantitive aspects of behaviour is taken into consideration, as well as functional correctness.
- Fluid approximation based analysis offers hope for scalable quantitative analysis techniques, but there remain many interesting and challenging problems to be solved.
- In particular we currently seek to bring the fluid approximation techniques to systems with distinct locations.

Thank you!

Thank you!

Thanks to the other members of the QUANTICOL project

quanticol

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