Fluid Approximation for the Analysis of Collective Adaptive Systems

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11th March 2015

Outline

1. Introduction
   - Collective Systems
   - Quantitative Analysis
   - Stochastic Process Algebra
2. Quantitative Analysis of Collective Systems
   - Model construction
   - Mathematical analysis: fluid approximation
   - Numerical illustration
   - Deriving properties: fluid model checking
3. Conclusions

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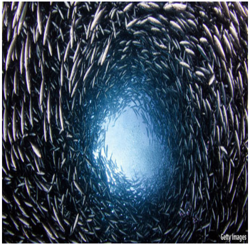

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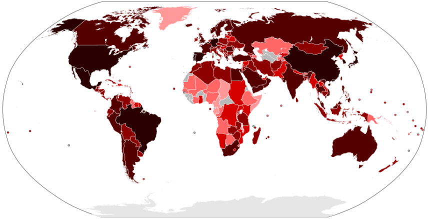

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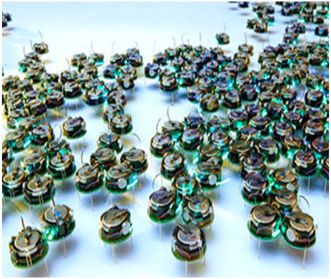

The Informatic Environment

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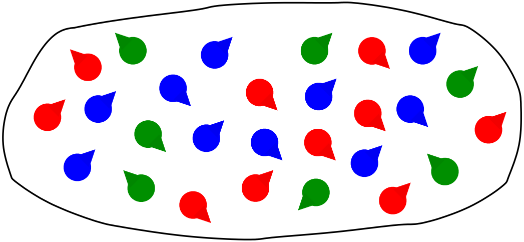
For instance, many examples of such systems can be found in components of Smart Cities, such as smart urban transport and smart grid electricity generation and storage.

Collective Systems

From a computer science perspective these systems can be viewed as being made up of a large number of interacting entities.

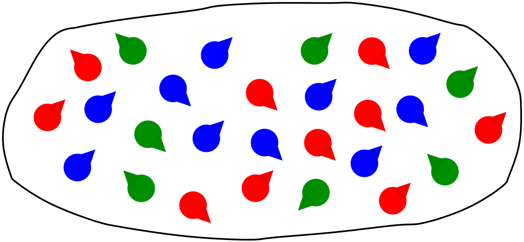


Each entity may have its own properties, objectives and actions.

At the system level these combine to create the collective behaviour.

Collective Systems

The behaviour of the system is thus dependent on the behaviour of the individual entities.



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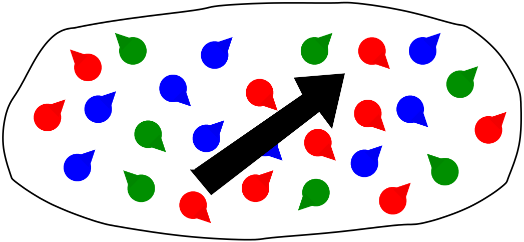

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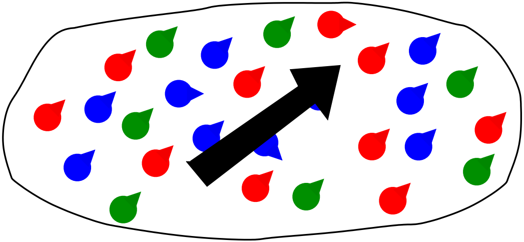

And the behaviour of the individuals will be influenced by the state of the overall system.

Quantitative Modelling

Performance modelling aims to construct models of the dynamic behaviour of systems in order to support the efficient and equitable sharing of resources.

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These techniques are no longer widely applicable for expressing the dynamic behaviour observed in distributed systems, and this is even more true of systems with collective behaviour.

Performance Modelling: Motivation

Capacity Planning

- How many clients can the existing server support and maintain reasonable response times?
- How many buses do I need to maintain service at peak time in a smart urban transport system?

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System Tuning

- In an automated factory what speed of conveyor belt will minimize robot idle time and jamming but maximize throughput?
- What strategy can I use to maintain supply-demand balance within a smart electricity grid?

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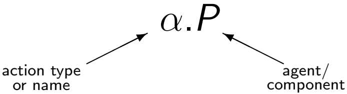
Primary examples include:

- Stochastic Petri Nets and
- Stochastic/Markovian Process Algebras.

Process Algebra

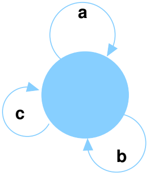
- Models consist of agents which engage in actions.



$\alpha.P$

action type or name

agent/ component



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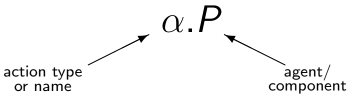

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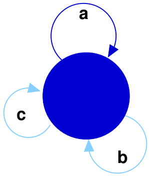

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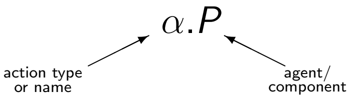

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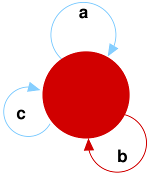

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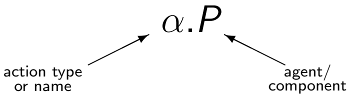

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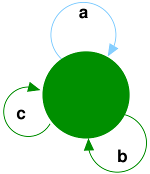

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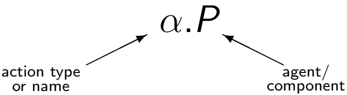

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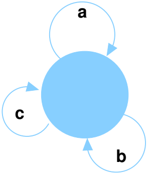

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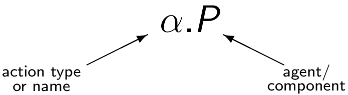

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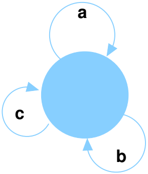

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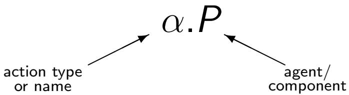
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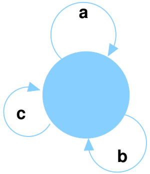

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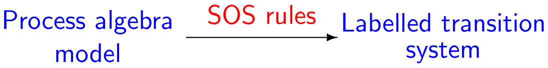
Process algebra model $\xrightarrow{\text{SOS rules}}$ Labelled transition system

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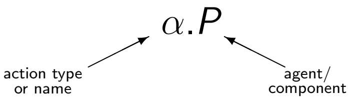

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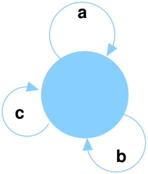

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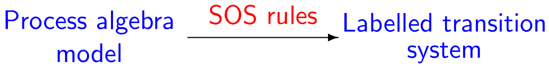

Process algebra model

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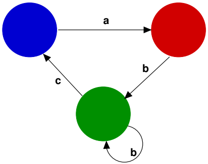
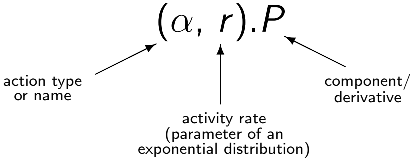


Stochastic process algebras

Process algebras where models are decorated with quantitative information used to generate a stochastic process are stochastic process algebras (SPA).

Stochastic Process Algebra

- Models are constructed from components which engage in activities.



$$(\alpha, r).P$$

action type or name

activity rate (parameter of an exponential distribution)

component/ derivative

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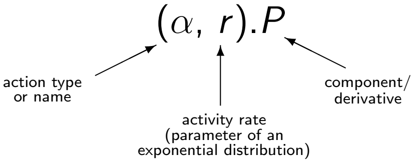

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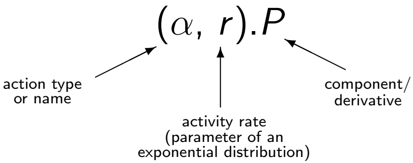

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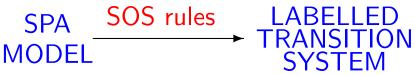

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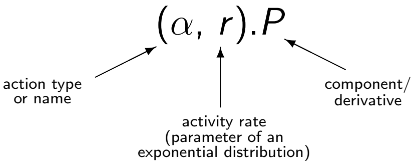

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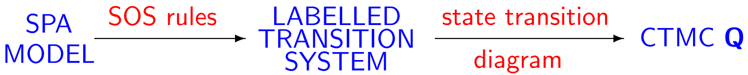

SPA MODEL

SOS rules

LABELLED TRANSITION SYSTEM

state transition diagram

CTMC **Q**

Integrated analysis

Qualitative verification can now be complemented by quantitative verification.

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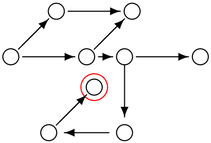
Reachability analysis

How long will it take for the system to arrive in a particular state?



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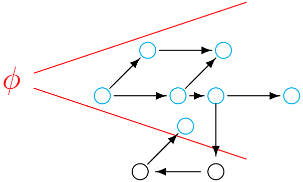
Model checking

Does a given property ϕ hold within the system with a given probability?

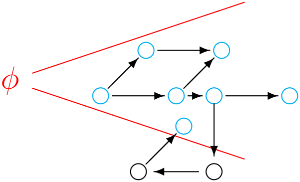


Integrated analysis

Qualitative verification can now be complemented by quantitative verification.

Model checking

For a given starting state
how long is it until
a given property ϕ holds?



Performance Evaluation Process Algebra

$(\alpha, f).P$	Prefix
$P_1 + P_2$	Choice
$P_1 \underset{L}{\bowtie} P_2$	Co-operation
P/L	Hiding
X	Constant

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When working with large numbers of entities, we write $P[n]$ to denote an array of n copies of P executing in parallel.

$$P[5] \equiv (P \parallel P \parallel P \parallel P \parallel P)$$

Structured Operational Semantics

PEPA is defined using a Plotkin-style structured operational semantics (a “small step” semantics).

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Choice

$$\frac{E \xrightarrow{(\alpha,r)} E'}{E + F \xrightarrow{(\alpha,r)} E'}$$

$$\frac{F \xrightarrow{(\alpha,r)} F'}{E + F \xrightarrow{(\alpha,r)} F'}$$

Structured Operational Semantics: Cooperation ($\alpha \notin L$)

Cooperation

$$\frac{E \xrightarrow{(\alpha,r)} E'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,r)} E' \underset{L}{\bowtie} F} \; (\alpha \notin L)$$

Structured Operational Semantics: Cooperation ($\alpha \notin L$)

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Structured Operational Semantics: Cooperation ($\alpha \in L$)

Cooperation $$\frac{E \xrightarrow{(\alpha,r_1)} E' \qquad F \xrightarrow{(\alpha,r_2)} F'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,R)} E' \underset{L}{\bowtie} F'} \; (\alpha \in L)$$

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Cooperation $$\frac{E \xrightarrow{(\alpha,r_1)} E' \qquad F \xrightarrow{(\alpha,r_2)} F'}{E \underset{L}{\bowtie} F \xrightarrow{(\alpha,R)} E' \underset{L}{\bowtie} F'} \ (\alpha \in L)$$

where $R = \dfrac{r_1}{r_\alpha(E)} \dfrac{r_2}{r_\alpha(F)} min(r_\alpha(E), r_\alpha(F))$

Apparent Rate

$$
\begin{aligned}
r_\alpha((\beta, r).P) &= \begin{cases} r & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases} \\
r_\alpha(P + Q) &= r_\alpha(P) + r_\alpha(Q) \\
r_\alpha(A) &= r_\alpha(P) \qquad \text{where } A \overset{def}{=} P \\
r_\alpha(P \underset{L}{\bowtie} Q) &= \begin{cases} r_\alpha(P) + r_\alpha(Q) & \alpha \notin L \\ min(r_\alpha(P), r_\alpha(Q)) & \alpha \in L \end{cases} \\
r_\alpha(P/L) &= \begin{cases} r_\alpha(P) & \alpha \notin L \\ 0 & \alpha \in L \end{cases}
\end{aligned}
$$

Structured Operational Semantics: Hiding

Hiding

$$\frac{E \xrightarrow{(\alpha,r)} E'}{E/L \xrightarrow{(\alpha,r)} E'/L} \; (\alpha \notin L)$$

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Structured Operational Semantics: Constants

Constant

$$\frac{E \xrightarrow{(\alpha,r)} E'}{A \xrightarrow{(\alpha,r)} E'} \; (A \overset{\mathit{def}}{=} E)$$

A simple example: processors and resources

$$
\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\overset{def}{=} (task1, r_3).Res_1 \\
Res_1 &\overset{def}{=} (reset, r_4).Res_0
\end{aligned}
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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

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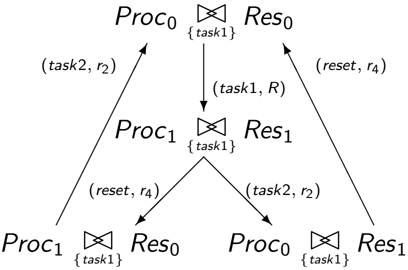

$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$

$(task2, r_2)$

$(task1, R)$

$(reset, r_4)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_1$

$(reset, r_4)$

$(task2, r_2)$

$Proc_1 \underset{\{task1\}}{\bowtie} Res_0$

$Proc_0 \underset{\{task1\}}{\bowtie} Res_1$

$$R = \min(r_1, r_3)$$

A simple example: processors and resources

$$\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
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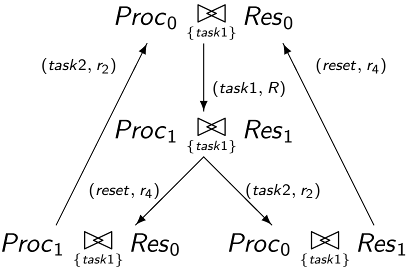
\end{aligned}$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$



$Proc_0 \underset{\{task1\}}{\bowtie} Res_0$

$(task2, r_2)$

$(task1, R)$

$(reset, r_4)$

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$$R = \min(r_1, r_3)$$

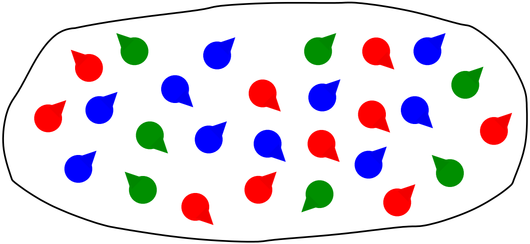
$$\mathbf{Q} = \begin{pmatrix} -R & R & 0 & 0 \\ 0 & -(r_2 + r_4) & r_4 & r_2 \\ r_2 & 0 & -r_2 & 0 \\ r_4 & 0 & 0 & -r_4 \end{pmatrix}$$

Outline

1. Introduction
   - Collective Systems
   - Quantitative Analysis
   - Stochastic Process Algebra
2. Quantitative Analysis of Collective Systems
   - Model construction
   - Mathematical analysis: fluid approximation
   - Numerical illustration
   - Deriving properties: fluid model checking
3. Conclusions

Modelling collective behaviour

A key feature of collective systems is the existence of populations of entities who share certain characteristics.



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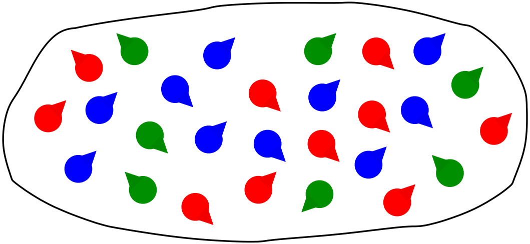

High-level modelling formalisms allow this repetition to be captured at the high-level rather than explicitly.

Process Algebra for Collective Systems

Process algebra are well-suited for constructing models of collective systems:

- Developed to represent concurrent behaviour compositionally;

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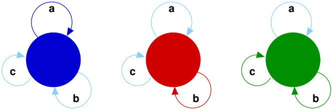
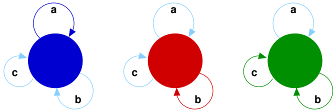
Recent advances in analysis techniques for process algebras have made it possible to study such systems even when the number of entities and activities become huge.

Solving discrete state models



Under the SOS semantics a SPA model is mapped to a CTMC with global states determined by the local states of all the participating components.

Solving discrete state models



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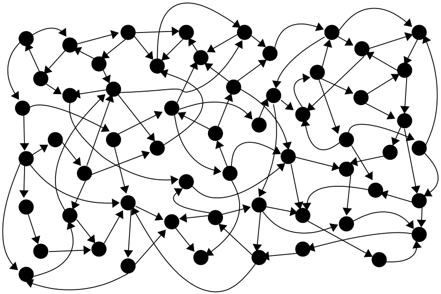

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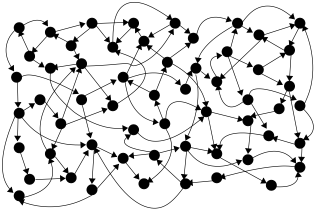

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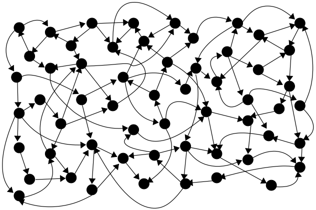

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$$Q = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,N} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,N} \\ \vdots & \vdots & & \vdots \\ q_{N,1} & q_{N,2} & \cdots & q_{N,N} \end{pmatrix}$$

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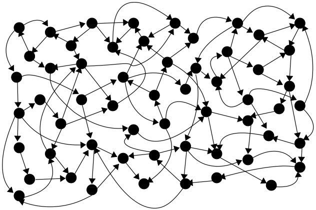

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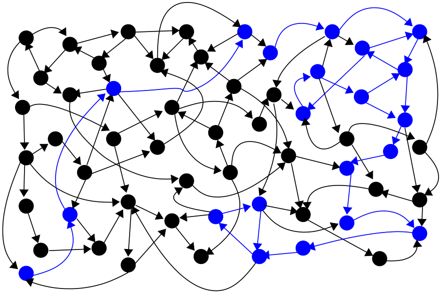
$$\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_N(t))$$

$$\pi(\infty)Q = 0$$

Solving discrete state models

Alternatively they may be studied using stochastic simulation. Each run generates a single trajectory through the state space. Many runs are needed in order to obtain average behaviours.



State space explosion

As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

Identity and Individuality

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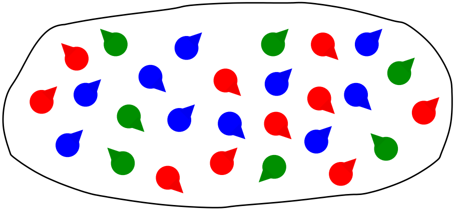

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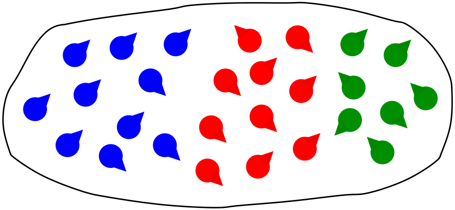

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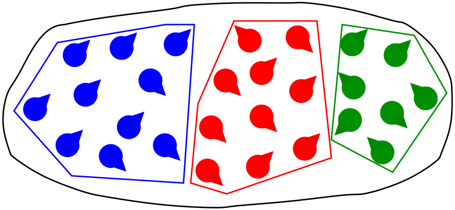

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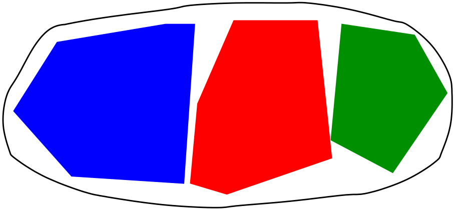

We may choose to disregard the identity of components.

Even better reductions can be achieved when we no longer regard the components as individuals.

Population statistics: emergent behaviour

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This allows us to model much larger systems than previously possible but in making the shift we are no longer able to collect any information about individuals in the system.

To characterise the behaviour of a population we calculate the proportion of individuals within the population that are exhibiting certain behaviours rather than tracking individuals directly.

Furthermore we make a continuous approximation of how the proportions vary over time.

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We use ordinary differential equations to represent the evolution of those variables over time.

Simple example revisited

$$
\begin{aligned}
Proc_0 &\stackrel{\mathit{def}}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{\mathit{def}}{=} (task2, r_2).Proc_0 \\
Res_0 &\stackrel{\mathit{def}}{=} (task1, r_3).Res_1 \\
Res_1 &\stackrel{\mathit{def}}{=} (reset, r_4).Res_0
\end{aligned}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

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\end{aligned}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

CTMC interpretation

Processors (N_P)	Resources (N_R)	States ($2^{N_P+N_R}$)
1	1	4
2	1	8
2	2	16
3	2	32
3	3	64
4	3	128
4	4	256
5	4	512
5	5	1024
6	5	2048
6	6	4096
7	6	8192
7	7	16384
8	7	32768
8	8	65536
9	8	131072
9	9	262144
10	9	524288
10	10	1048576

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\end{aligned}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

- $task1$ decreases $Proc_0$ and Res_0
- $task1$ increases $Proc_1$ and Res_1
- $task2$ decreases $Proc_1$
- $task2$ increases $Proc_0$
- $reset$ decreases Res_1
- $reset$ increases Res_0

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$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\min(r_1\, x_1, r_3\, x_3) + r_2\, x_2 \\
x_1 &= \text{no. of } Proc_1
\end{aligned}
$$

- *task*1 decreases $Proc_0$
- *task*1 is performed by $Proc_0$ and Res_0
- *task*2 increases $Proc_0$
- *task*2 is performed by $Proc_1$

Simple example revisited

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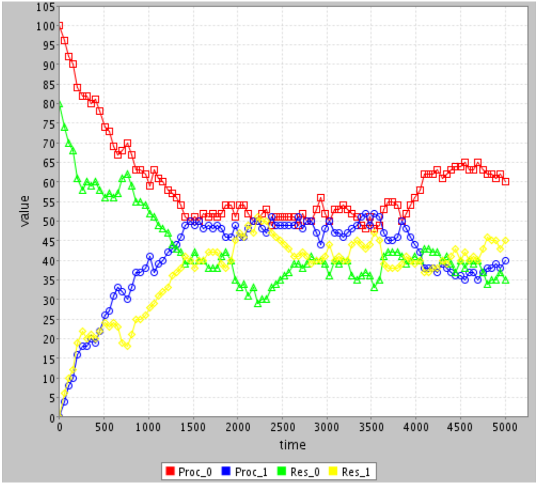
\end{aligned}
$$

$$Proc_0[N_P] \underset{\{task1\}}{\bowtie} Res_0[N_R]$$

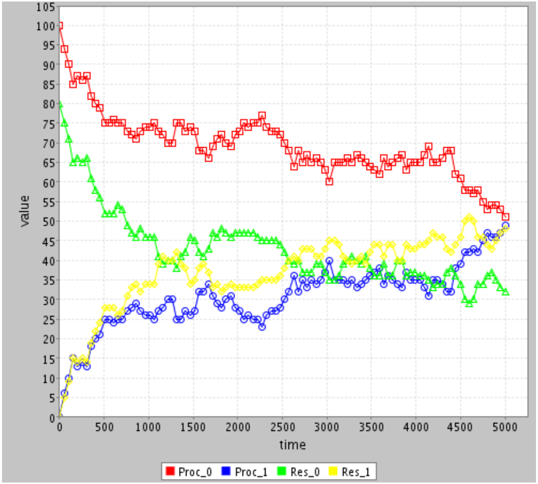
ODE interpretation

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\min(r_1\, x_1, r_3\, x_3) + r_2\, x_2 \\
& \quad x_1 = \text{no. of } Proc_1 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \min(r_1\, x_1, r_3\, x_3) - r_2\, x_2 \\
& \quad x_2 = \text{no. of } Proc_2 \\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\min(r_1\, x_1, r_3\, x_3) + r_4\, x_4 \\
& \quad x_3 = \text{no. of } Res_0 \\
\frac{\mathrm{d}x_4}{\mathrm{d}t} &= \min(r_1\, x_1, r_3\, x_3) - r_4\, x_4 \\
& \quad x_4 = \text{no. of } Res_1
\end{aligned}
$$

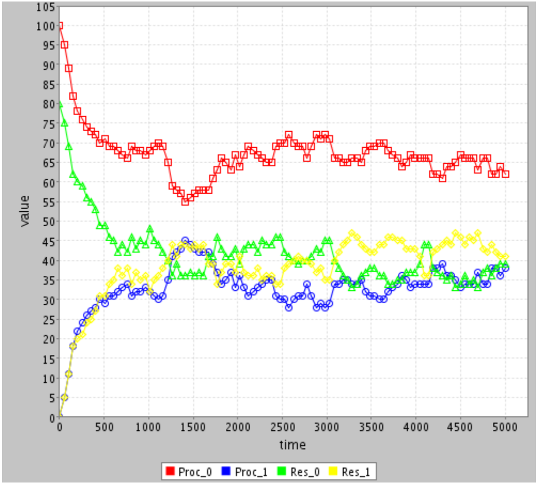
100 processors and 80 resources (simulation run A)



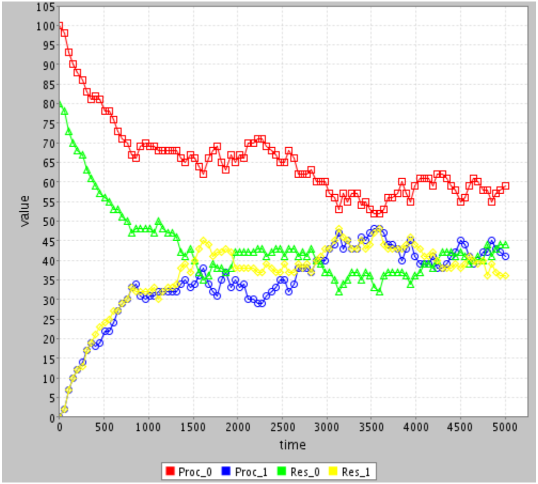
100 processors and 80 resources (simulation run B)



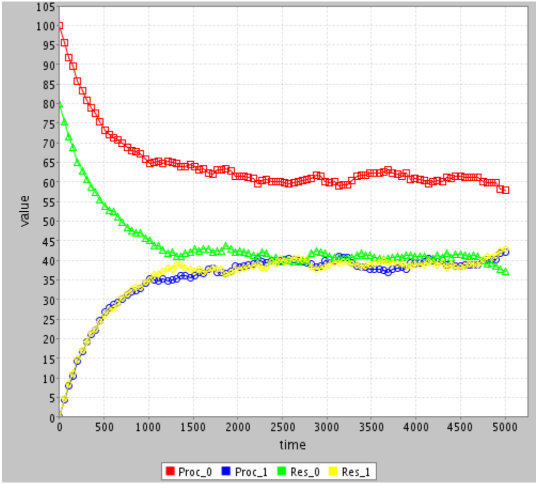
100 processors and 80 resources (simulation run C)



100 processors and 80 resources (simulation run D)



100 processors and 80 resources (average of 10 runs)



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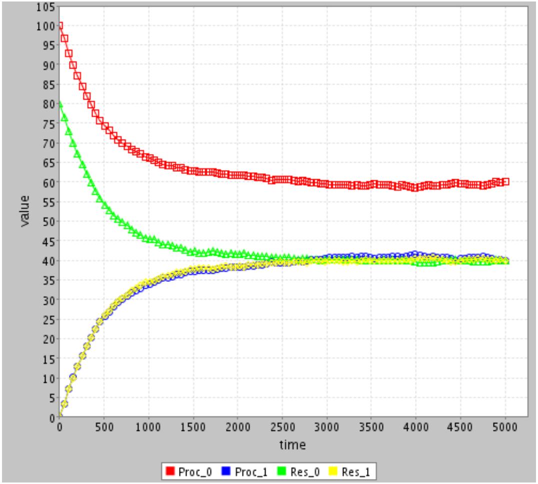
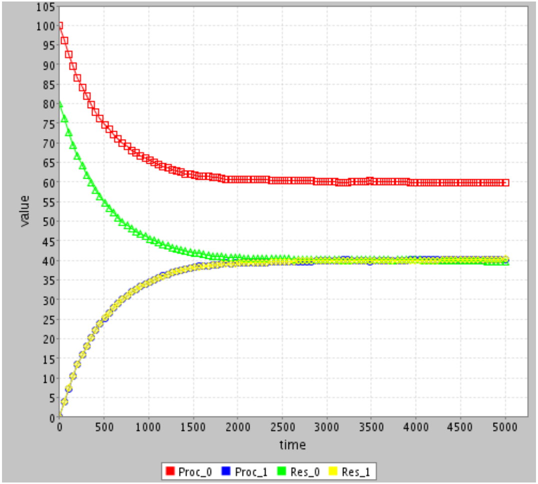
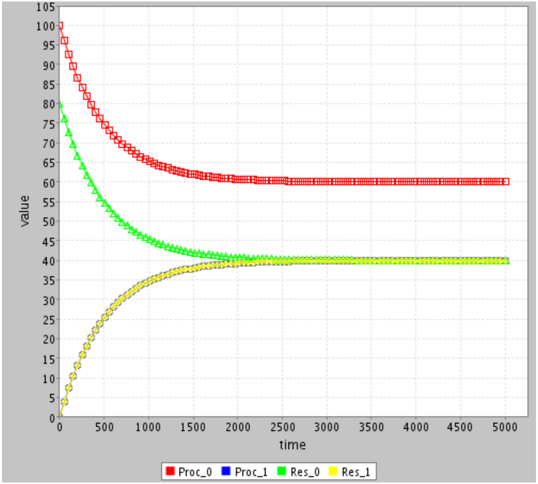


100 processors and 80 resources (average of 1000 runs)



100 processors and 80 resources (ODE solution)



Deriving a Fluid Approximation of a SPA model

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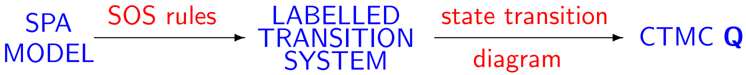

SPA MODEL

SOS rules

LABELLED TRANSITION SYSTEM

state transition

diagram

CTMC **Q**

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We define a structured operational semantics which defines the possible transitions of an arbitrary abstract state and from this derive the ODEs.

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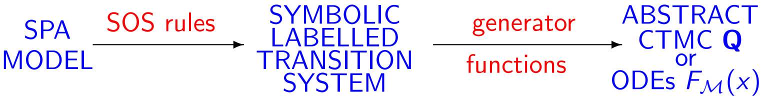
SPA MODEL
SOS rules
SYMBOLIC LABELLED TRANSITION SYSTEM
generator
functions
ABSTRACT CTMC $\mathbf{Q}$
or
ODEs $F_{\mathcal{M}}(x)$

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2. Collect the transitions of the reduced context as symbolic updates on the state representation (Jump Multiset)
3. Calculate the rate of the transitions in terms of an arbitrary state of the CTMC.

Once this is done we can extract the vector field $F_{\mathcal{M}}(x)$ from the jump multiset, under the assumption that the population size tends to infinity.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

Context Reduction

$$\begin{aligned}
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Res_1 &\stackrel{def}{=} (reset, r_4).Res_0 \\
System &\stackrel{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}$$

$$\Downarrow$$

$$\mathcal{R}(System) = \{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\}$$

Context Reduction

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System &\stackrel{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}$$

$$\Downarrow$$

$$\mathcal{R}(System) = \{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\}$$

Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

Location Dependency

$$System \overset{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

Location Dependency

$$System \stackrel{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

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Location Dependency

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$$\Downarrow$$

$$\{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\} \parallel \{Proc_0, Proc_1\}$$

Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

Fluid Structured Operational Semantics by Example

$$\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1\\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0\\
Res_0 &\overset{def}{=} (task1, r_3).Res_1\\
Res_1 &\overset{def}{=} (reset, r_4).Res_0\\
System &\overset{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

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\end{aligned}$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\frac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1}}{}$$

Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
Res_0 &\overset{def}{=} (task1, r_3).Res_1 \\
Res_1 &\overset{def}{=} (reset, r_4).Res_0 \\
System &\overset{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$
\dfrac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{}
$$

Fluid Structured Operational Semantics by Example

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System &\stackrel{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}$$

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\dfrac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_{*} Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_{*} Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

Apparent Rate Calculation

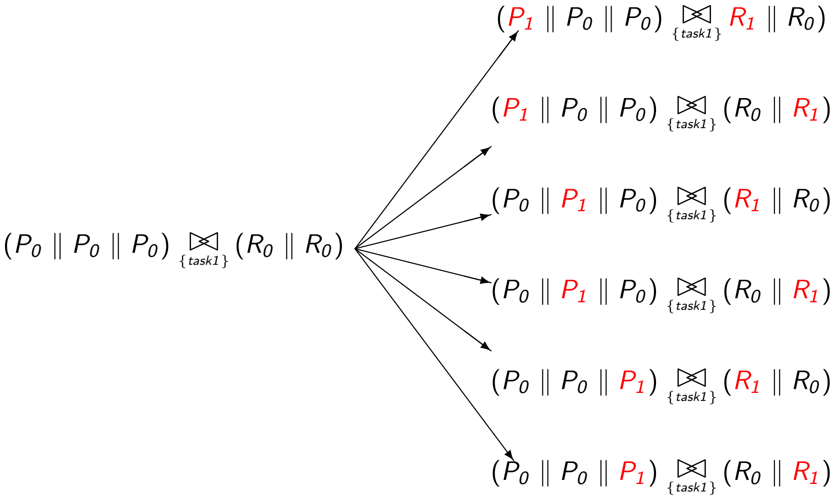
$$\frac{\dfrac{\mathit{Proc}_0 \xrightarrow{\mathit{task1}, r_1} \mathit{Proc}_1}{\mathit{Proc}_0 \xrightarrow{\mathit{task1}, r_1\xi_1}_* \mathit{Proc}_1} \qquad \dfrac{\mathit{Res}_0 \xrightarrow{\mathit{task1}, r_3} \mathit{Res}_1}{\mathit{Res}_0 \xrightarrow{\mathit{task1}, r_3\xi_3}_* \mathit{Res}_1}}{\mathit{Proc}_0 \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res}_0 \xrightarrow{\mathit{task1}, r(\xi)}_* \mathit{Proc}_1 \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res}_1}$$

Apparent Rate Calculation

$$\dfrac{\dfrac{Proc_0 \xrightarrow{task1,r_1} Proc_1}{Proc_0 \xrightarrow{task1,r_1\xi_1}_{*} Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1,r_3} Res_1}{Res_0 \xrightarrow{task1,r_3\xi_3}_{*} Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_4\right)$$

$f(\xi, I, \alpha)$ as the Generator Matrix of the *Lumped* CTMC



$(P_0 \parallel P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0)$

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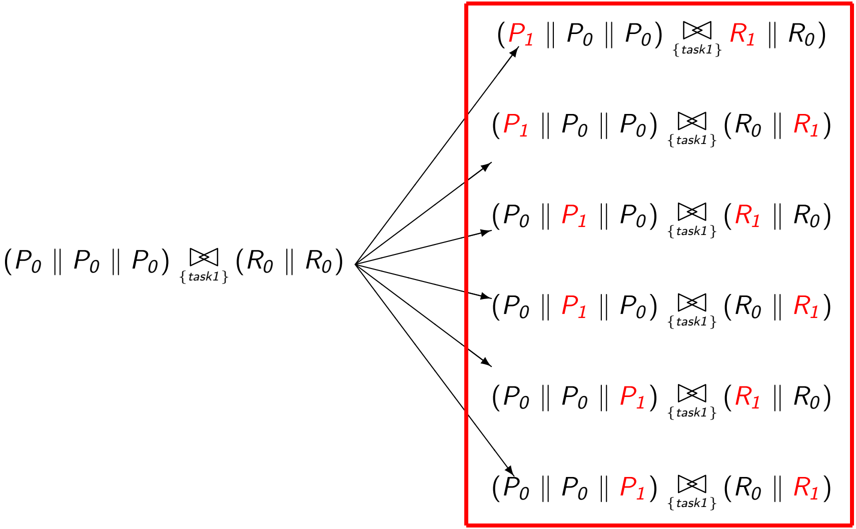

$(P_0 \parallel P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0)$

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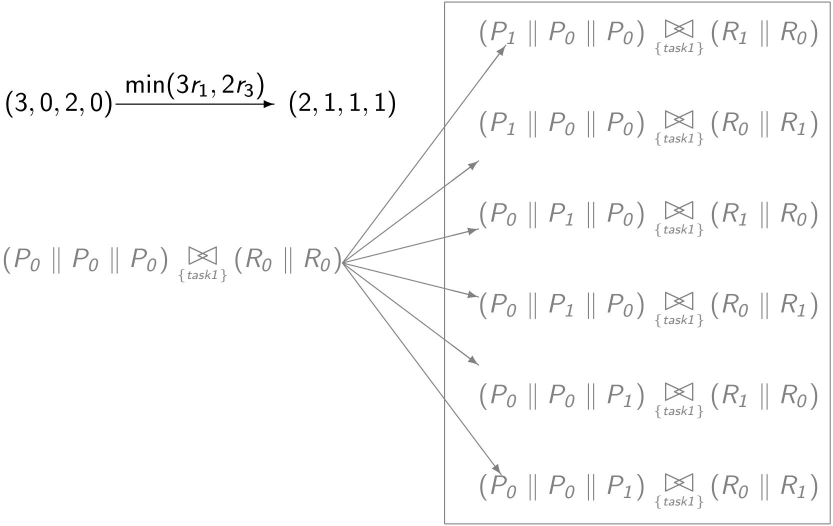
$f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(3,0,2,0) \xrightarrow{\min(3r_1, 2r_3)} (2,1,1,1)$$



$(P_0 \parallel P_0 \parallel P_0) \bowtie_{\{task1\}} (R_0 \parallel R_0)$

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$(P_0 \parallel P_0 \parallel P_1) \bowtie_{\{task1\}} (R_0 \parallel R_1)$

Jump Multiset

$$\mathit{Proc_0} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_0} \xrightarrow{\mathit{task1},\, r(\xi)}_{*} \mathit{Proc_1} \underset{\{\mathit{task1}\}}{\bowtie} \mathit{Res_1}$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2,\ \xi_2 r_2}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2,\ \xi_2 r_2}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Equivalent Transitions

Some transitions may give the same information:

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$
$$Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\, \xi_4 r_4}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_0$$

i.e., Res_1 may perform an action independently from the rest of the system.

This is captured by the procedure used for the construction of the generator function $f(\xi, l, \alpha)$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$

Construction of $f(\xi, I, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\, \xi_4 r_4}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $I = (0, 0, 0, 0)$
- Add -1 to all elements of I corresponding to the indices of the components in the lhs of the transition

$$I = (-1, 0, 0, -1)$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$
- Add -1 to all elements of l corresponding to the indices of the components in the lhs of the transition

$$l = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of l corresponding to the indices of the components in the rhs of the transition

$$l = (-1 + 1, 0, +1, -1) = (0, 0, +1, -1)$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$
- Add -1 to all elements of l corresponding to the indices of the components in the lhs of the transition

$$l = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of l corresponding to the indices of the components in the rhs of the transition

$$l = (-1 + 1, 0, +1, -1) = (0, 0, +1, -1)$$

$$f\left(\xi, (0, 0, +1, -1), reset\right) = \xi_4 r_4$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2,\ \xi_2 r_2'}_{*} Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$\begin{aligned} f(\xi, (-1, +1, -1, +1), task1) &= r(\xi) \\ f(\xi, (+1, -1, 0, 0), task2) &= \xi_2 r_2 \end{aligned}$$

Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,\ r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$
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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\ \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$\begin{aligned} f(\xi, (-1, +1, -1, +1), task1) &= r(\xi) \\ f(\xi, (+1, -1, 0, 0), task2) &= \xi_2 r_2 \\ f(\xi, (0, 0, +1, -1), reset) &= \xi_4 r_4 \end{aligned}$$

Capturing behaviour in the Generator Function

$$
\begin{aligned}
Proc_0 &\overset{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\overset{def}{=} (task2, r_2).Proc_0 \\
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System &\overset{def}{=} Proc_0[N_P] \underset{\{transfer\}}{\bowtie} Res_0[N_R]
\end{aligned}
$$

Capturing behaviour in the Generator Function

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\end{array}$$

Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \ \text{ and } \ \xi_3 + \xi_4 = N_R$$

Capturing behaviour in the Generator Function

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\end{array}$$

Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \text{ and } \xi_3 + \xi_4 = N_R$$

Generator Function

$$f(\xi, l, \alpha): \quad \begin{array}{rcl}
f(\xi, (-1, 1, -1, 1), task1) & = & \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1, -1, 0, 0), task2) & = & r_2\xi_2 \\
f(\xi, (0, 0, 1, -1), reset) & = & r_4\xi_4
\end{array}$$

Extraction of the ODE from f

Generator Function

$$\begin{aligned} f(\xi, (-1,1,-1,1), \mathit{task1}) &= \min(r_1\xi_1, r_3\xi_3) \\ f(\xi, (1,-1,0,0), \mathit{task2}) &= r_2\xi_2 \\ f(\xi, (0,0,1,-1), \mathit{reset}) &= r_4\xi_4 \end{aligned}$$

Differential Equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_{\mathcal{M}}(x) = \sum_{l \in \mathbb{Z}^d} l \sum_{\alpha \in \mathcal{A}} f(x, l, \alpha)$$

$$= (-1,1,-1,1)\min(r_1x_1, r_3x_3) + (1,-1,0,0)r_2x_2 + (0,0,1,-1)r_4x_4$$

Extraction of the ODE from f

Generator Function

$$\begin{aligned} f(\xi,(-1,1,-1,1),\mathit{task1}) &= \min(r_1\xi_1, r_3\xi_3) \\ f(\xi,(1,-1,0,0),\mathit{task2}) &= r_2\xi_2 \\ f(\xi,(0,0,1,-1),\mathit{reset}) &= r_4\xi_4 \end{aligned}$$

Differential Equation

$$\begin{aligned} \frac{dx_1}{dt} &= -\min(r_1x_1, r_3x_3) + r_2x_2 \\ \frac{dx_2}{dt} &= \min(r_1x_1, r_3x_3) - r_2x_2 \\ \frac{dx_3}{dt} &= -\min(r_1x_1, r_3x_3) + r_4x_4 \\ \frac{dx_4}{dt} &= \min(r_1x_1, r_3x_3) - r_4x_4 \end{aligned}$$

Consistency results

- The vector field $\mathcal{F}(x)$ is Lipschitz continuous i.e. all the rate functions governing transitions in the process algebra satisfy local continuity conditions.

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  Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes, T.G. Kurtz, J. Appl. Prob. (1970).

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- The generated ODEs are the fluid limit of the family of CTMCs: this family forms a sequence as the initial populations are scaled by a variable n.
- We can prove this using Kurtz's theorem: Solutions of Ordinary Differential Equations as Limits of Pure Jump Markov Processes, T.G. Kurtz, J. Appl. Prob. (1970).
- Moreover Lipschitz continuity of the vector field guarantees existence and uniqueness of the solution to the initial value problem.

M.Tribastone, S.Gilmore and J.Hillston. Scalable Differential Analysis of Process Algebra Models. IEEE TSE 2012.

Quantitative properties

The derived vector field $\mathcal{F}(x)$, gives an approximation of the expected count for each population over time.

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This has been extended in a number of ways:

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- Fluid approximation of passage times have been defined.

R.A.Hayden, A.Stefanek and J.T.Bradley. Fluid computation of passage-time distributions in large Markov models. TCS 2012.

Fluid model checking

Since the vector field records only deterministic behaviour, LTL model checking can be used over a trace to give boolean results.

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Work on this is on-going but there are initial results for:

- CSL properties of a single agent within a population.

L.Bortolussi and J.Hillston. Fluid model checking. CONCUR 2012.
L.Bortolussi and J.Hillston. Model checking single agent behaviour by fluid approximation. Inf & Comp 2015.

- The fraction of a population that satisfies a property expressed as a one-clock deterministic timed automaton.

L.Bortolussi and R.Lanciani. Central Limit Approximation for Stochastic Model Checking. QEST 2013.

CSL model checking of a single agent

We consider properties of a single agent within a population, expressed in the Continuous Stochastic Logic (CSL), usually used for model checking CTMCs.

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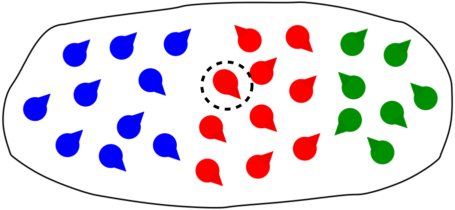
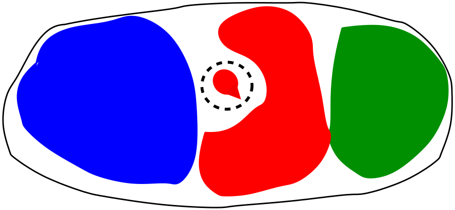
We consider an arbitrary member of the population.



CSL model checking of a single agent

We consider properties of a single agent within a population, expressed in the Continuous Stochastic Logic (CSL), usually used for model checking CTMCs.

This agent is kept discrete, making transitions between its discrete states, but all other agents are treated as a mean-field influencing the behaviour of this agent.



CSL model checking of a single agent

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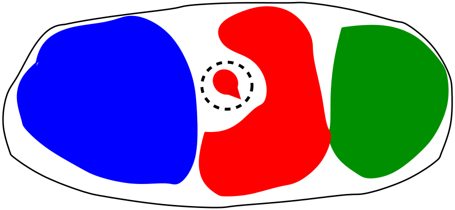
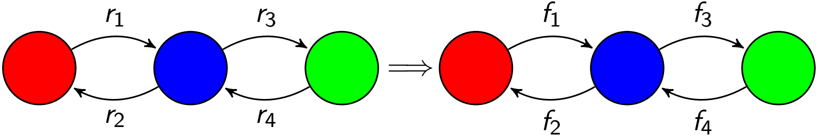
Essentially we keep a detailed discrete-event representation of the one agent and make a fluid approximation of the rest of the population.

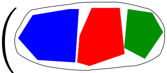


Inhomogeneous CTMC

The transition rates within the discrete-event representation will depend on the rest of the population.

i.e. it will depend on the vector field capturing the behaviour of the residual population.



where $f_i = f\Bigg($  $\Bigg)$

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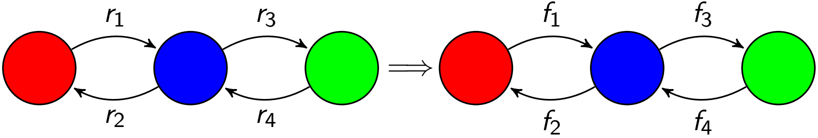
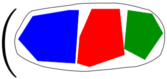
The transition rates within the discrete-event representation will depend on the rest of the population.

i.e. it will depend on the vector field capturing the behaviour of the residual population.



where $f_i = f\Big($  $\Big)$

It is an inhomogeneous continuous time Markov chain.

Model checking the ICTMC

Care is needed to model check the ICTMC, which proceeds by explicitly calculating the reachability probabilities for states of interest (analogously to CSL model checking on CTMCs).

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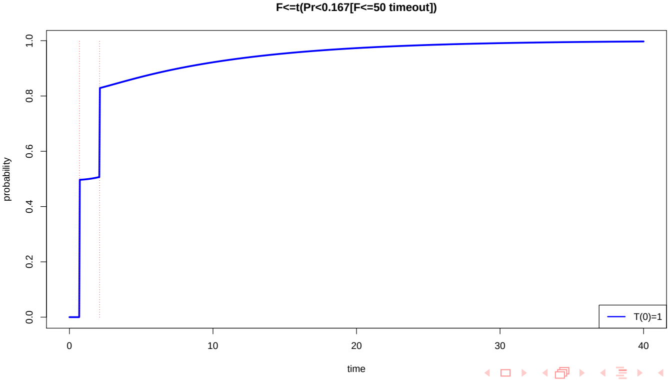
The inhomogeneous time within the model means that truth values may change with respect to time.



Outline

1. Introduction
   - Collective Systems
   - Quantitative Analysis
   - Stochastic Process Algebra
2. Quantitative Analysis of Collective Systems
   - Model construction
   - Mathematical analysis: fluid approximation
   - Numerical illustration
   - Deriving properties: fluid model checking
3. Conclusions

Conclusions

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- Their role within infrastructure, such as within smart cities, make it essential that quantitive aspects of behaviour is taken into consideration, as well as functional correctness.
- Fluid approximation based analysis offers hope for scalable quantitative analysis techniques, but there remain many interesting and challenging problems to be solved.
- In particular we currently seek to bring the fluid approximation techniques to systems with distinct locations.

Thank you!

Thank you!

Thanks to the other members of the QUANTICOL project

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