# Process algebras for quantitative analysis: Lecture 5 — Collective dynamics

Jane Hillston

School of Informatics
The University of Edinburgh
Scotland

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# Outline

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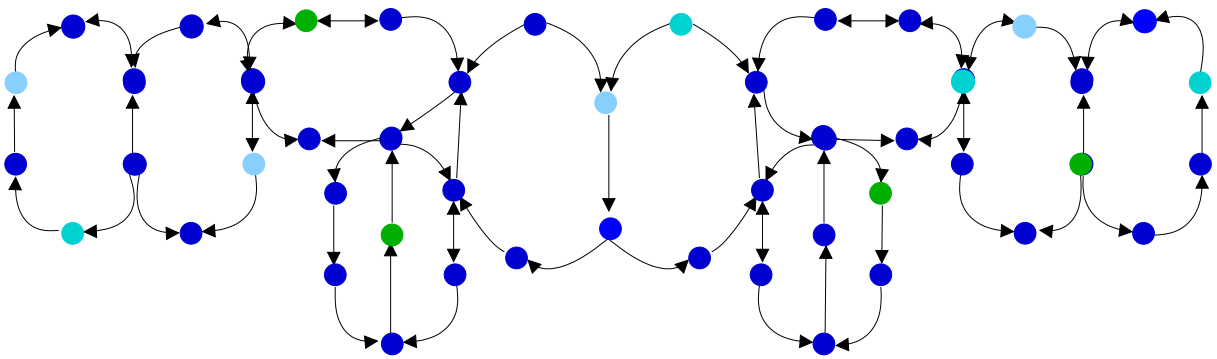
# Collective Dynamics

The behaviour of many systems can be interpreted as the result of the collective behaviour of a large number of interacting entities.



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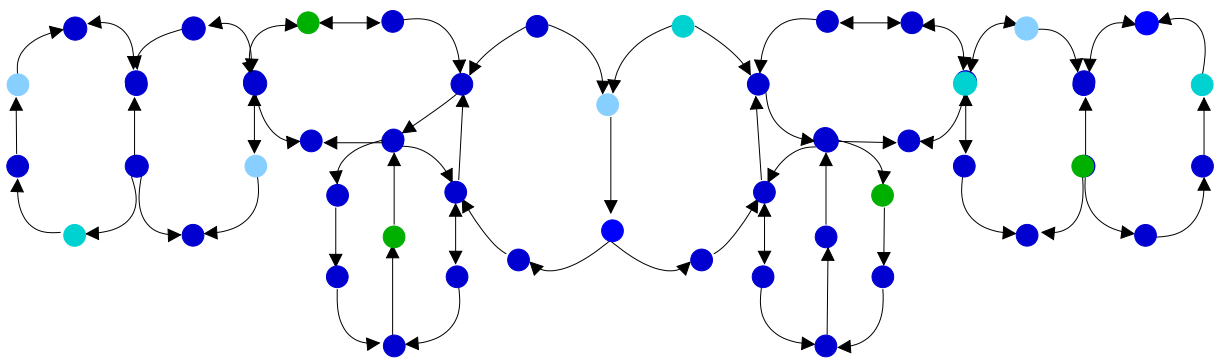

For such systems we are often as interested in the population level behaviour as we are in the behaviour of the individual entities.

# Collective Behaviour

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- Stochastic extensions, such as PEPA, enable quantified behaviour of the dynamics of systems.

In the CODA project we are developing stochastic process algebras and associated theory, tailored to the construction and evaluation of the collective dynamics of large systems of interacting entities.

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This allows us to model much larger systems than previously possible but in making the shift we are no longer able to collect any information about individuals in the system.

# Performance as an emergent behaviour

We must instead think about the performance of the collective point of view. Service providers often want to do this in any case. For example making contracts in terms of service level agreements.

**Example Service Level Agreement**

90% of requests receive a response within 3 seconds.

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**Example Service Level Agreement**

90% of requests receive a response within 3 seconds.

**Qualitative Service Level Agreement**

Less than 1% of the responses received within 3 seconds will read "System is overloaded, try again later".

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**Large scale software systems**
Issues of scalability are important for user satisfaction and resource efficiency but such issues are difficult to investigate using discrete state models.

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**Biochemical signalling pathways**
Understanding these pathways has the potential to improve the quality of life through enhanced drug treatment and better drug design.

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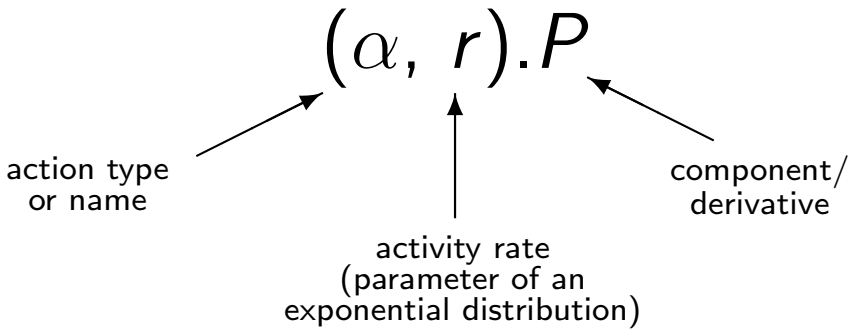
- The prospect of formally-based quantified evaluation of dynamic behaviour could have significant impact in application domains such as:

**Epidemiological systems**
Improved modelling of these systems could lead to improved disease prevention and treatment in nature and better security in computer systems.

# Stochastic Process Algebra

- Models are constructed from components which engage in activities.



# A simple example: processors and resources

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Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
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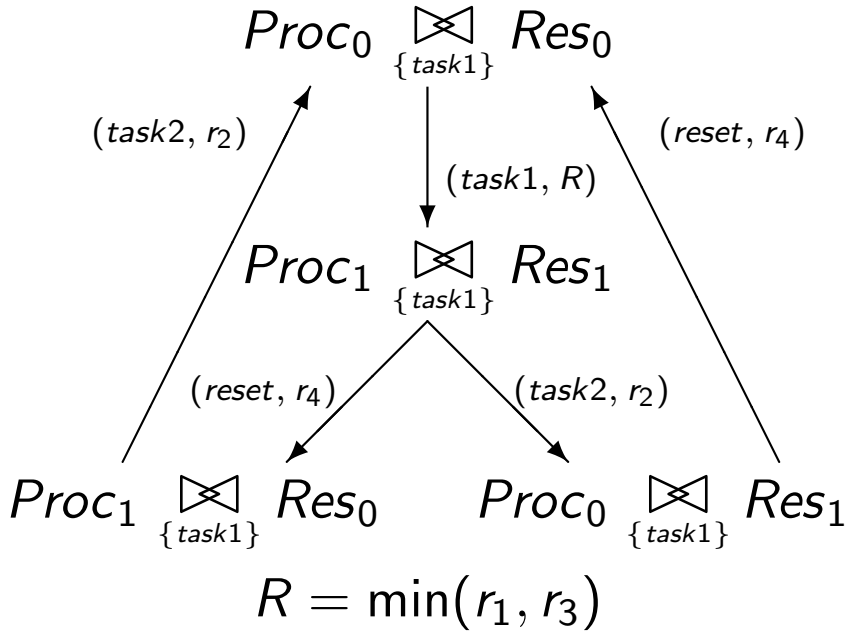

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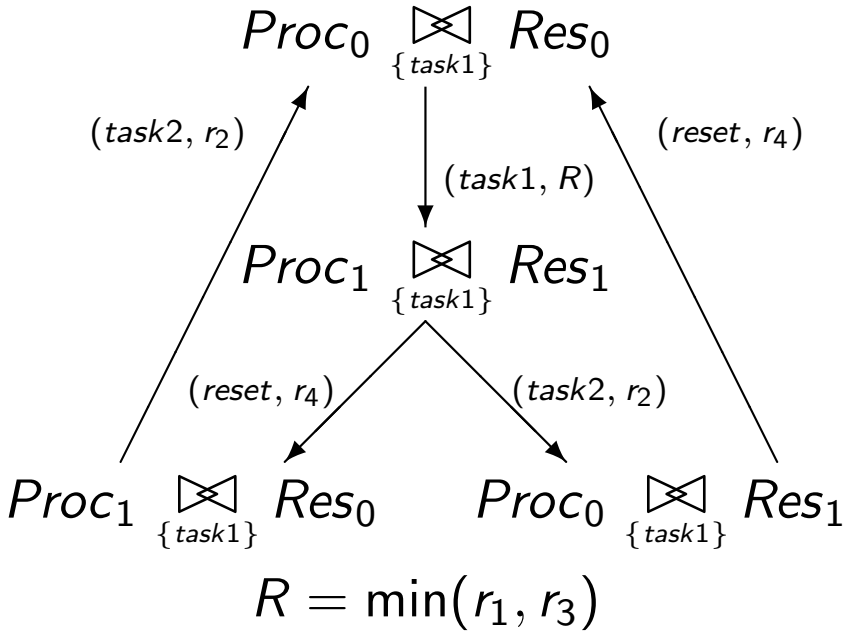


$$\mathbf{Q} = \begin{pmatrix} -R & R & 0 & 0 \\ 0 & -(r_2 + r_4) & r_4 & r_2 \\ r_2 & 0 & -r_2 & 0 \\ r_4 & 0 & 0 & -r_4 \end{pmatrix}$$

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Each of these has tool support so that the underlying model is derived automatically according to the predefined rules.

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# Solving discrete state models

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As the size of the state space becomes large it becomes infeasible to carry out numerical solution and extremely time-consuming to conduct stochastic simulation.

# Continuous Approximation

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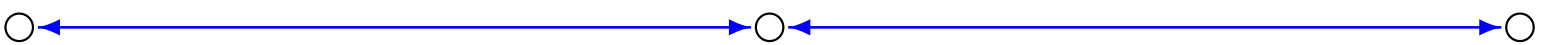

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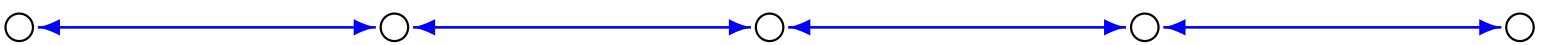

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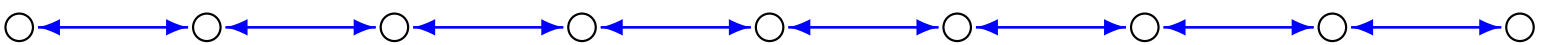

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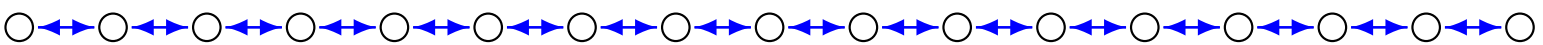

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Use ordinary differential equations to represent the evolution of those variables over time.

# New mathematical structures: differential equations

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- Assume that these state variables are subject to continuous rather than discrete change.
- No longer aim to calculate the probability distribution over the entire state space of the model.

Appropriate for models in which there are large numbers of components of the same type, i.e. models of populations and situations of collective dynamics.

# Simple example revisited [QEST'05]

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CTMC interpretation

| Processors ($N_P$) | Resources ($N_R$) | States ($2^{N_P+N_R}$) |
|---|---|---|
| 1 | 1 | 4 |
| 2 | 1 | 8 |
| 2 | 2 | 16 |
| 3 | 2 | 32 |
| 3 | 3 | 64 |
| 4 | 3 | 128 |
| 4 | 4 | 256 |
| 5 | 4 | 512 |
| 5 | 5 | 1024 |
| 6 | 5 | 2048 |
| 6 | 6 | 4096 |
| 7 | 6 | 8192 |
| 7 | 7 | 16384 |
| 8 | 7 | 32768 |
| 8 | 8 | 65536 |
| 9 | 8 | 131072 |
| 9 | 9 | 262144 |
| 10 | 9 | 524288 |
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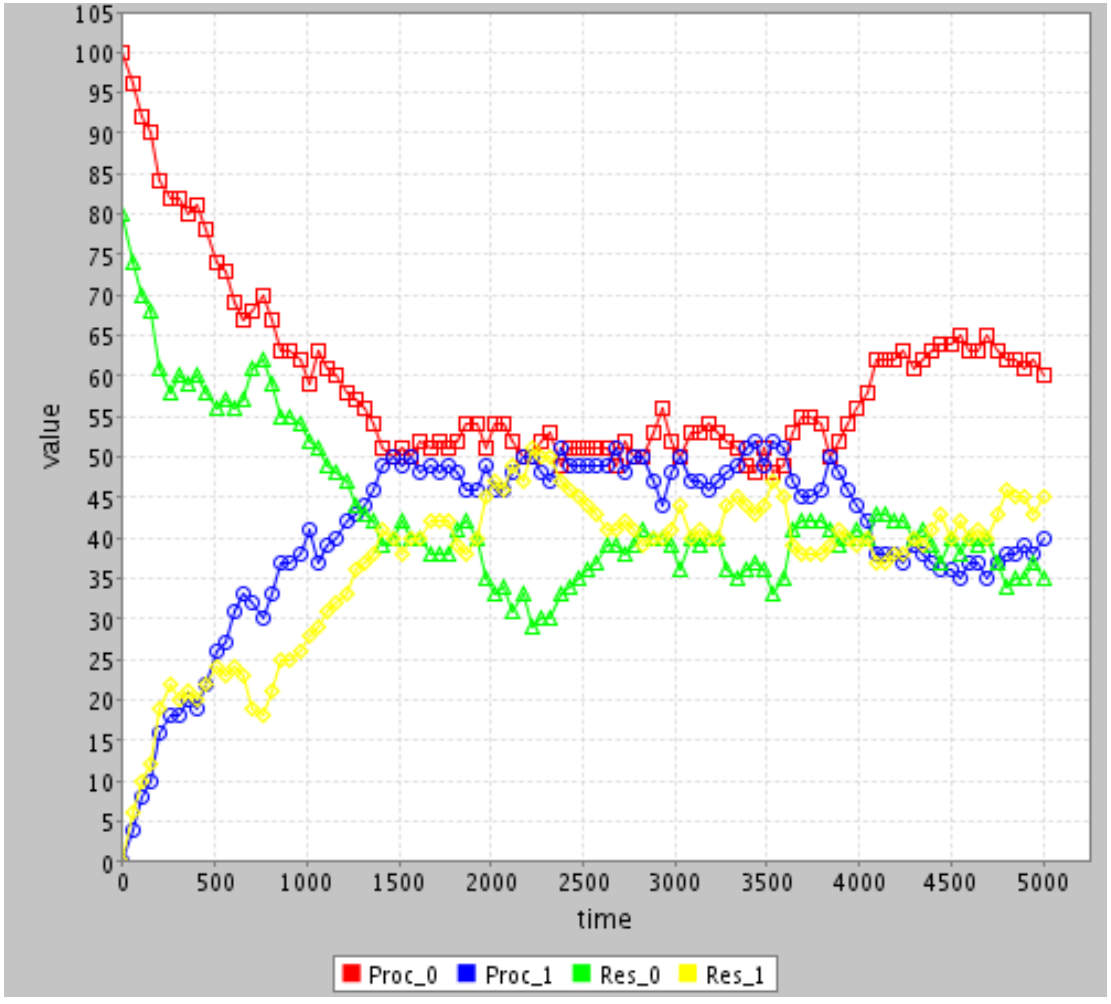
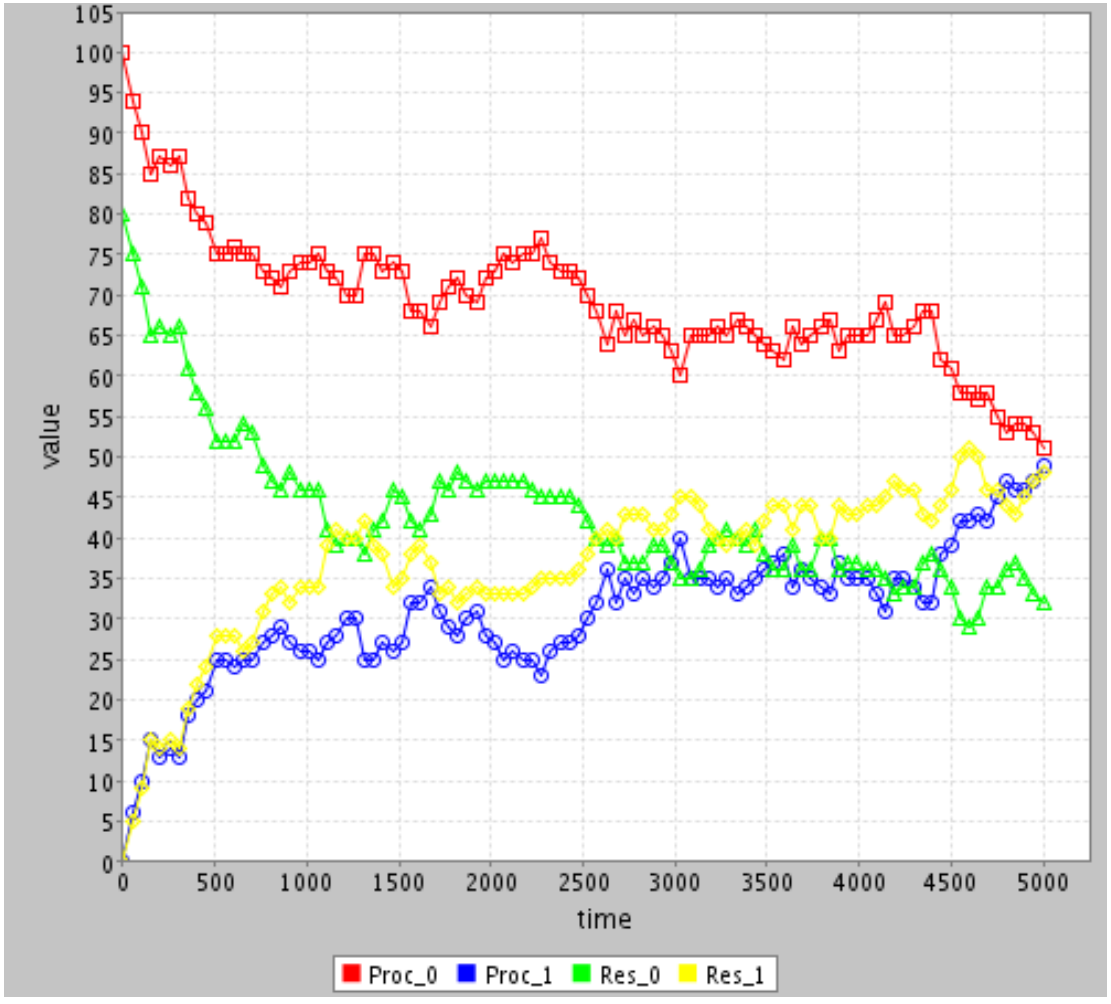
ODE interpretation

$$
\begin{aligned}
\frac{dx_1}{dt} &= -r_1 \min(x_1, x_3) + r_2 x_1 \\
&\quad x_1 = \text{no. of } Proc_1 \\
\frac{dx_2}{dt} &= r_1 \min(x_1, x_3) - r_2 x_1 \\
&\quad x_2 = \text{no. of } Proc_2 \\
\frac{dx_3}{dt} &= -r_1 \min(x_1, x_3) + r_4 x_4 \\
&\quad x_3 = \text{no. of } Res_0 \\
\frac{dx_4}{dt} &= r_1 \min(x_1, x_3) - r_4 x_4 \\
&\quad x_4 = \text{no. of } Res_1
\end{aligned}
$$

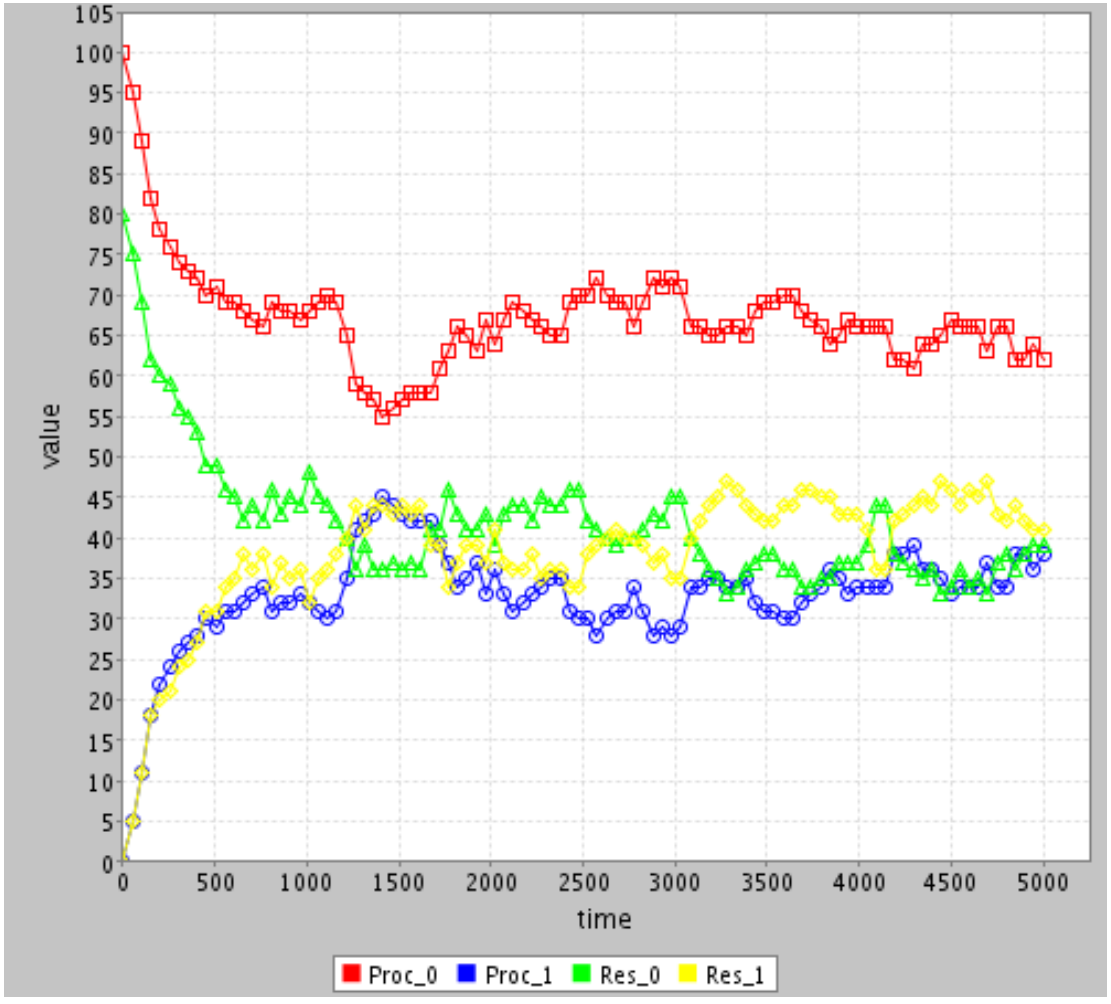
# 100 processors and 80 resources (simulation run A)



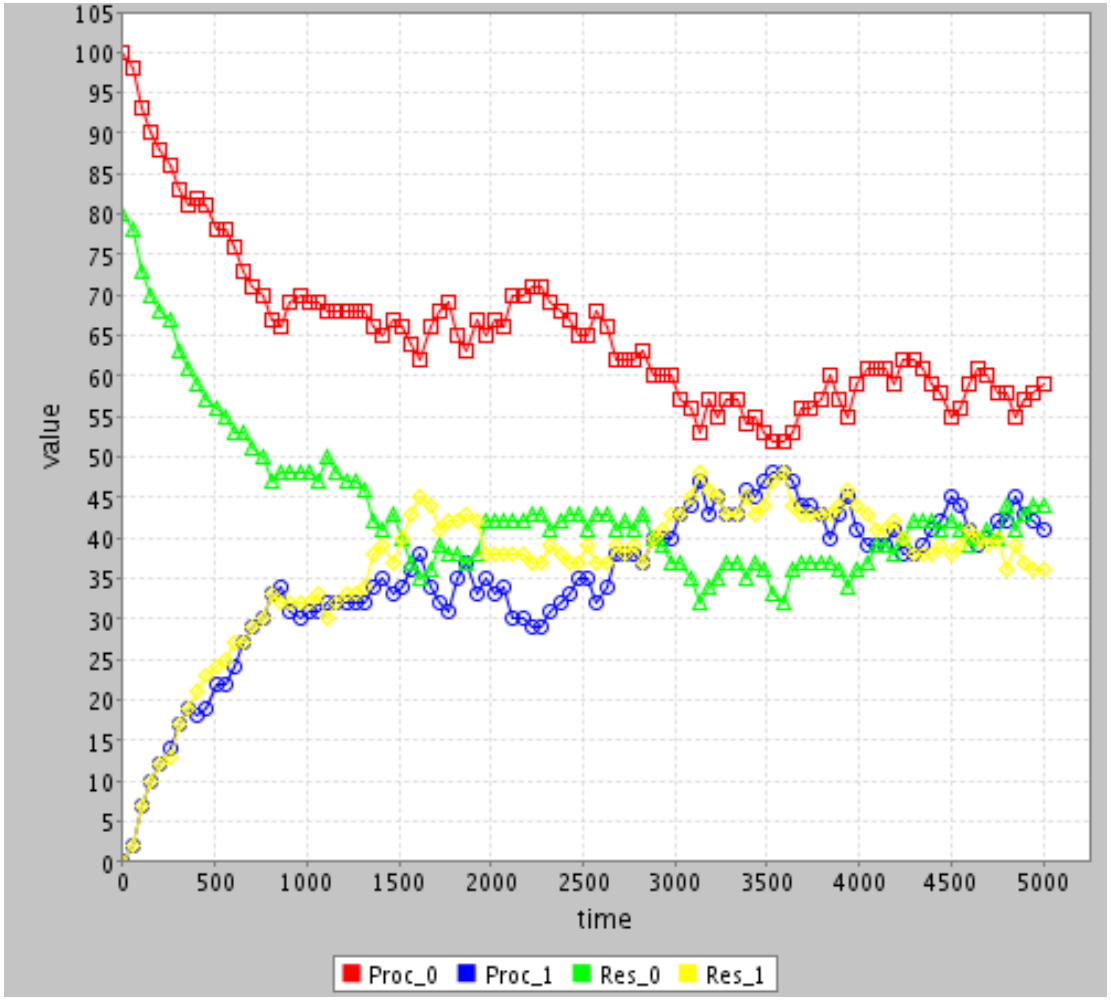
# 100 processors and 80 resources (simulation run B)



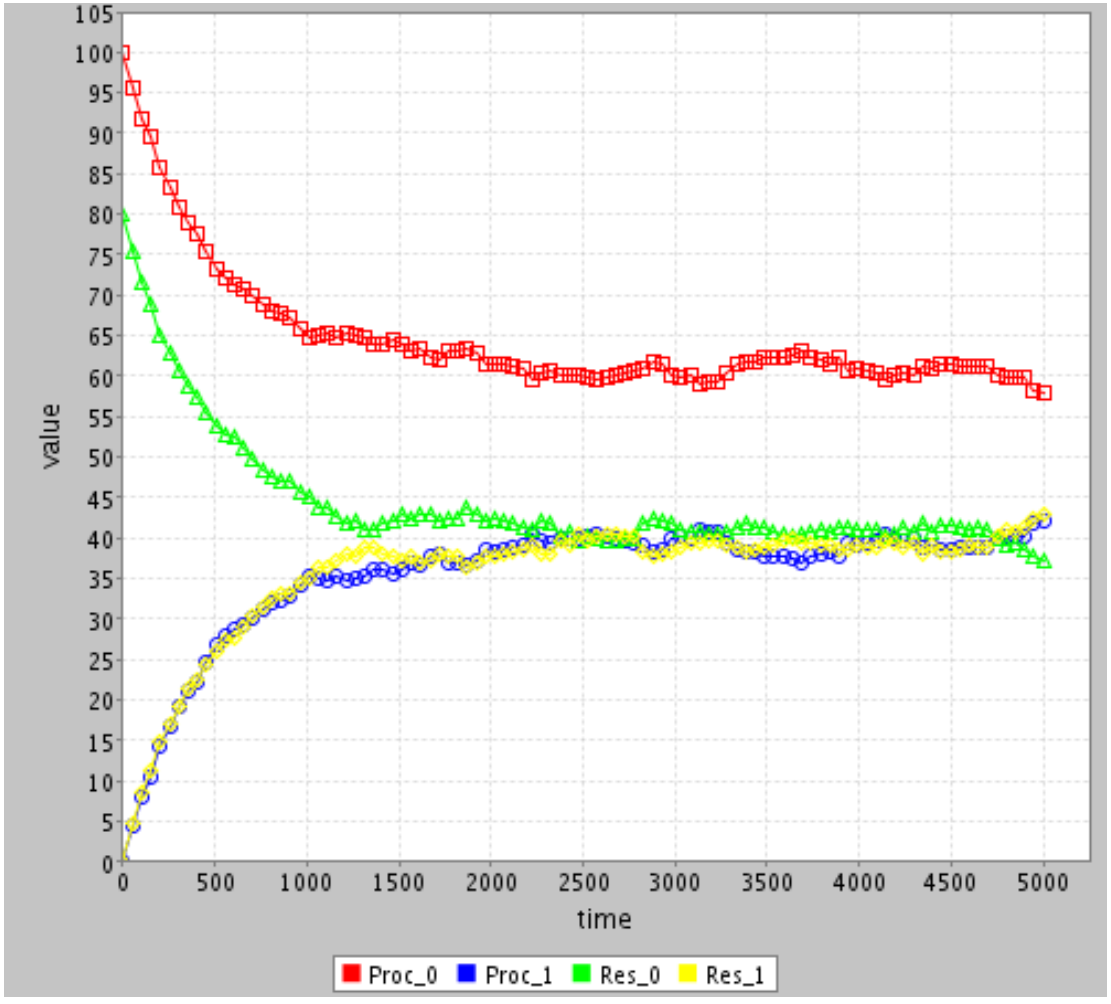
# 100 processors and 80 resources (simulation run C)



# 100 processors and 80 resources (simulation run D)



# 100 processors and 80 resources (average of 10 runs)



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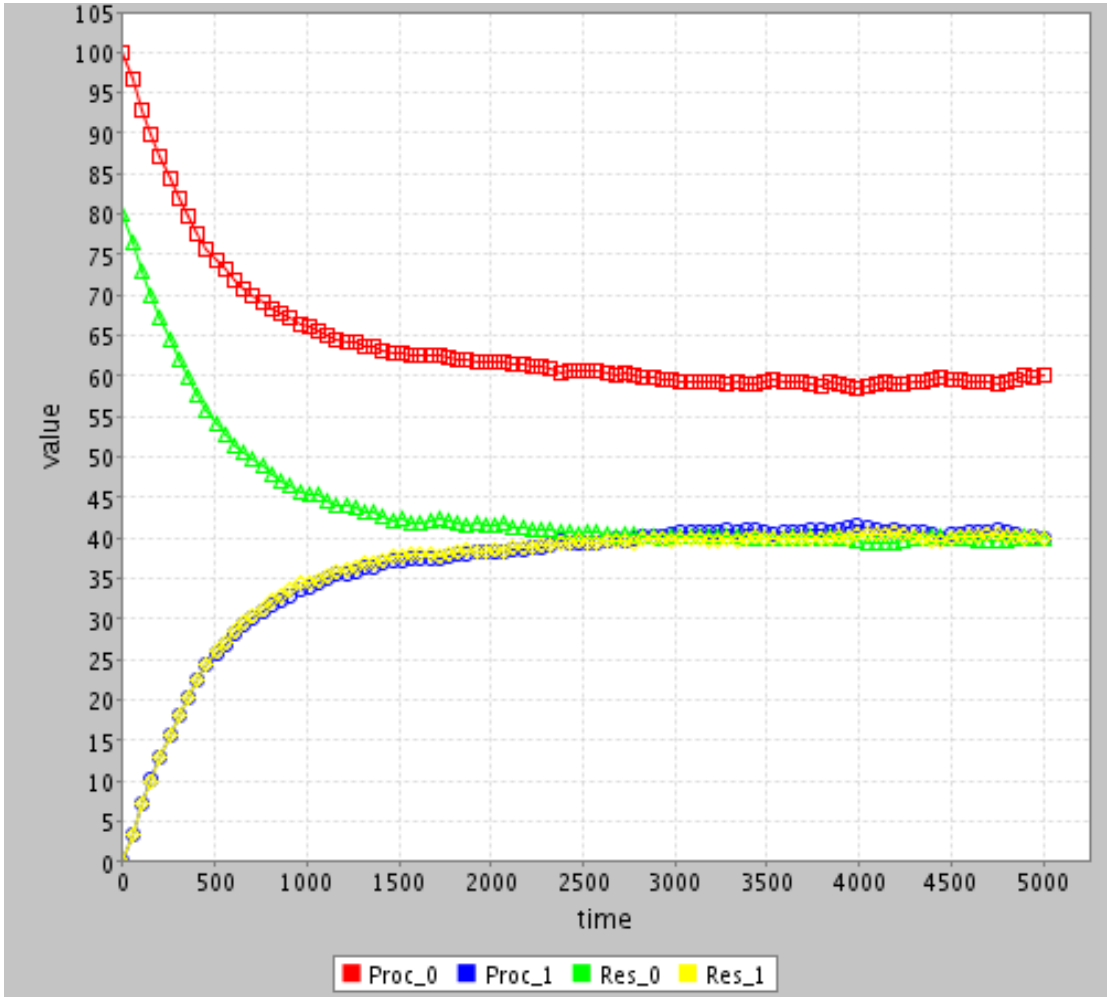

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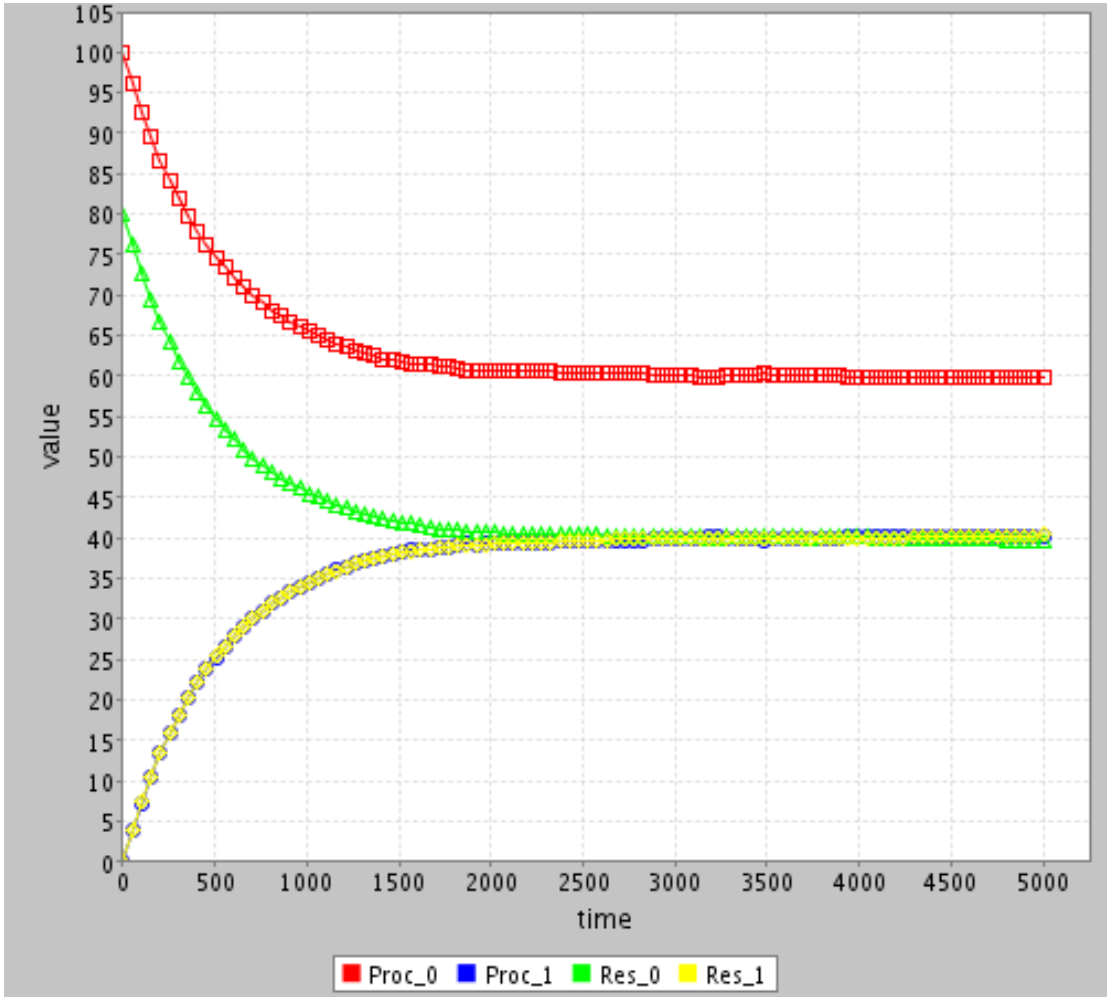
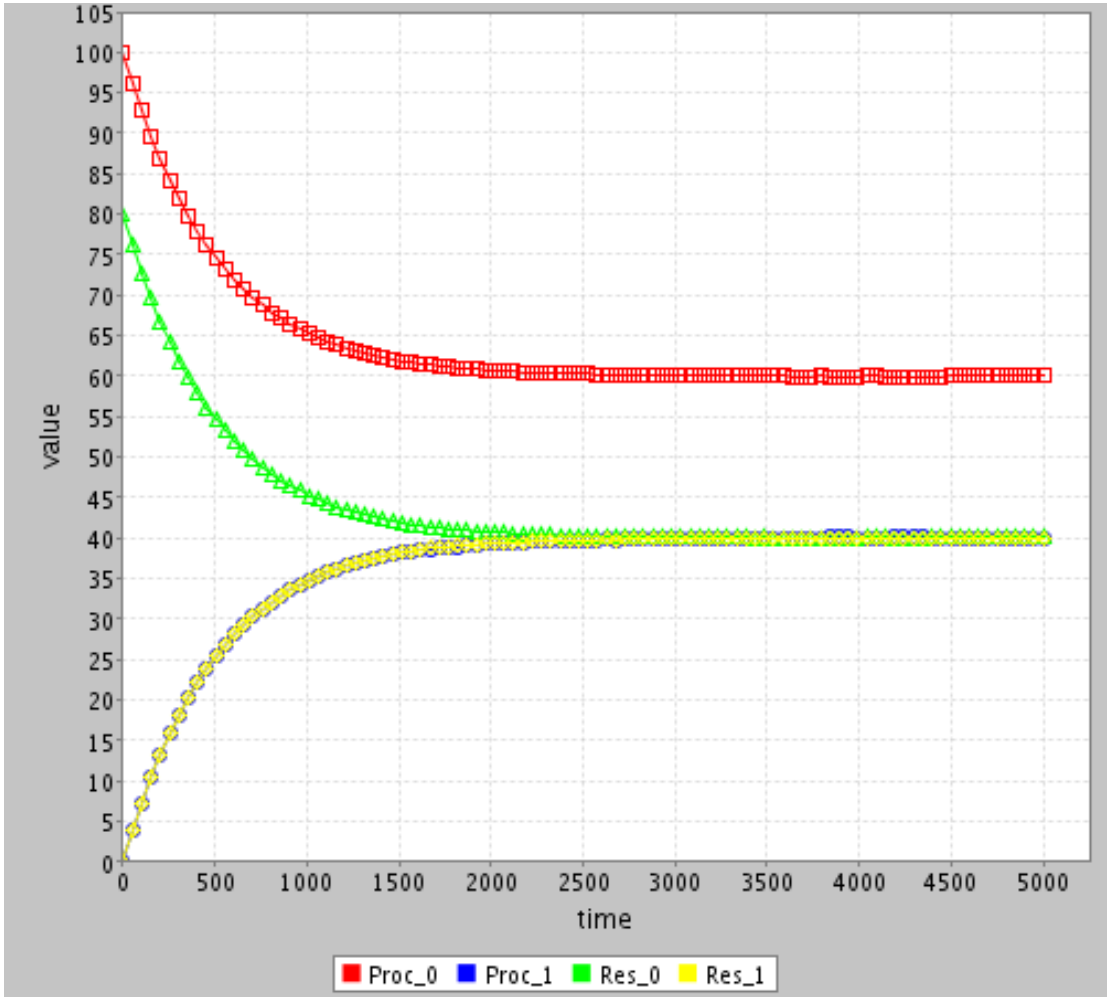
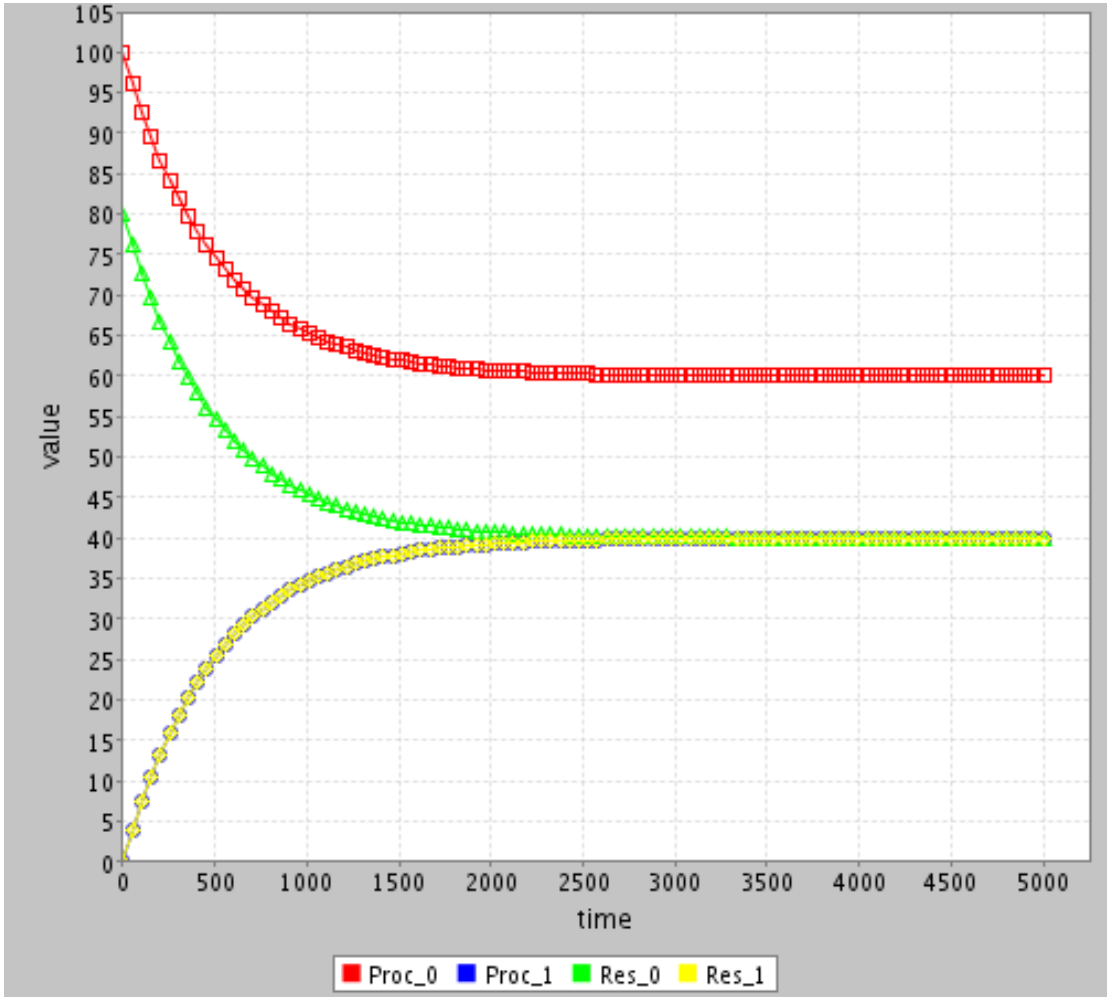


# 100 processors and 80 resources (average of 10000 runs)



# 100 processors and 80 resources (ODE solution)



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# Deriving a Fluid Approximation of a PEPA model

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The exisiting (CTMC) SOS semantics is not suitable for this purpose because it constructs the state space of the CTMC explicitly.

Nevertheless we are able to define a structured operational semantics which defines the possible transitions of an arbitrary abstract state and from this derive the ODEs.

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1. Remove excess components (*Context Reduction*)
2. Collect the transitions of the reduced context (*Jump Multiset*)
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Once this is done we can extract the vector field $F_{\mathcal{M}}(x)$ from the jump multiset.

# Context Reduction

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## Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

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$$System \stackrel{def}{=} Proc_0[N'_C] \underset{\{task1\}}{\bowtie} Res_0[N_S] \parallel Proc_0[N''_C]$$

$$\Downarrow$$

$$\{Proc_0, Proc_1\} \underset{\{task1\}}{\bowtie} \{Res_0, Res_1\} \parallel \{Proc_0, Proc_1\}$$

## Population Vector

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

# Fluid Structured Operational Semantics by Example

$$
\begin{aligned}
Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
Proc_1 &\stackrel{def}{=} (task2, r_2).Proc_0 \\
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$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$\frac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1}$$

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$$
\frac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1}
\qquad
\frac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}
$$

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$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$$

$$
\frac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}
$$

# Apparent Rate Calculation

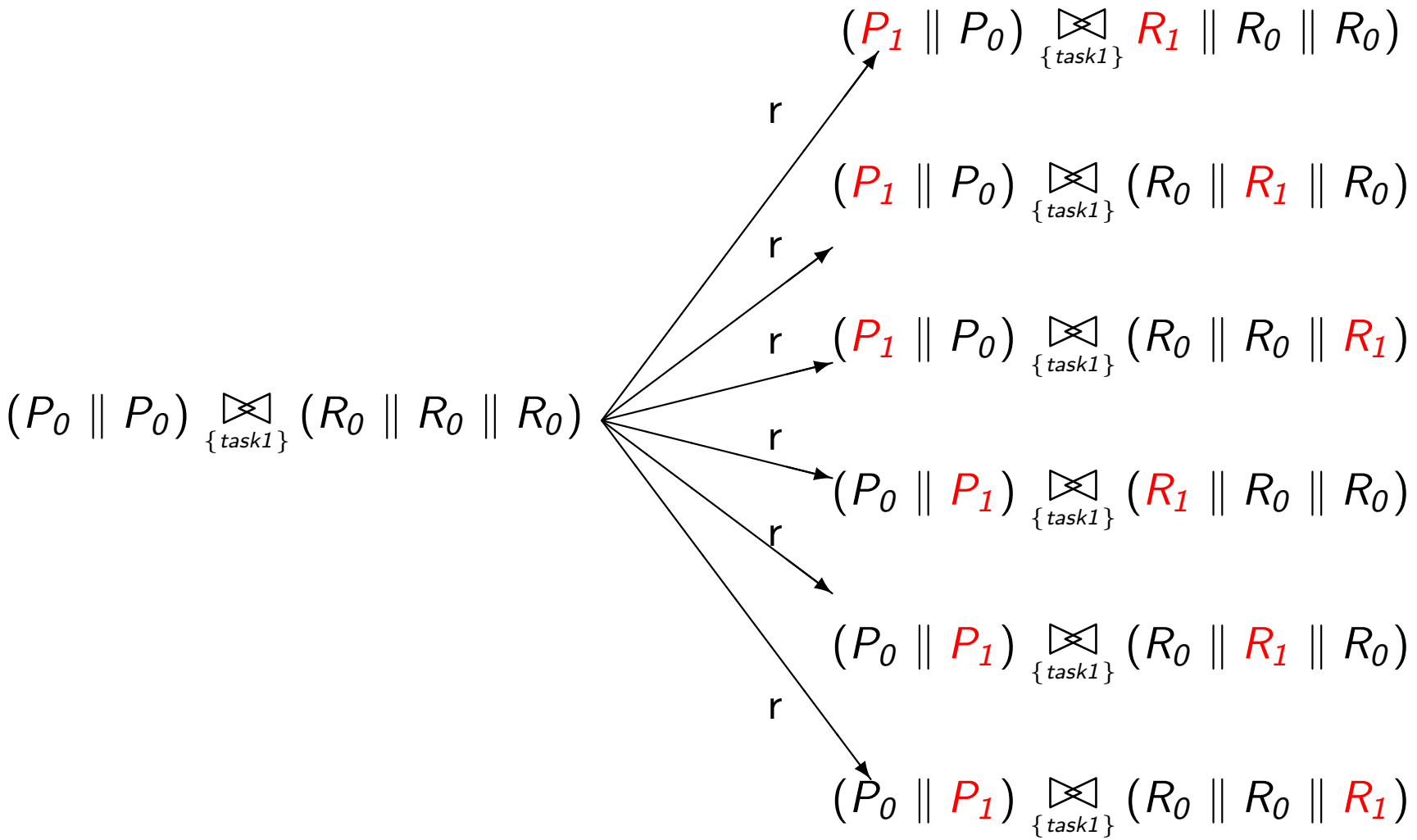
$$\frac{\dfrac{Proc_0 \xrightarrow{task1,r_1} Proc_1}{Proc_0 \xrightarrow{task1,r_1\xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1,r_3} Res_1}{Res_0 \xrightarrow{task1,r_3\xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1,r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

# Apparent Rate Calculation

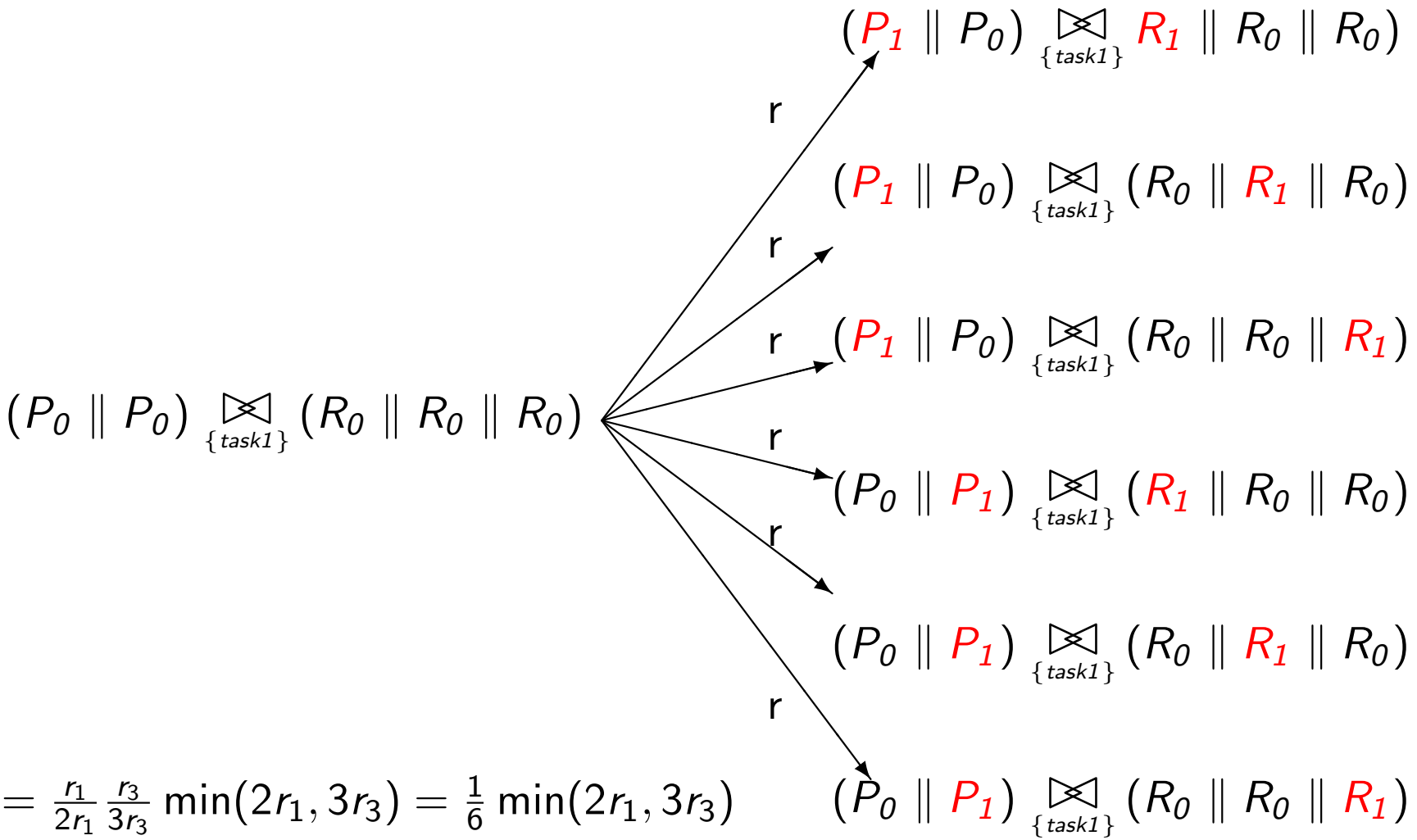
$$\frac{\dfrac{Proc_0 \xrightarrow{task1, r_1} Proc_1}{Proc_0 \xrightarrow{task1, r_1 \xi_1}_* Proc_1} \qquad \dfrac{Res_0 \xrightarrow{task1, r_3} Res_1}{Res_0 \xrightarrow{task1, r_3 \xi_3}_* Res_1}}{Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1}$$

$$\begin{aligned} r(\xi) &= \frac{r_1 \xi_1}{r^*_{task1}(Proc_0, \xi)} \frac{r_3 \xi_4}{r^*_{task1}(Res_0, \xi)} \min\left(r^*_{task1}(Proc_0, \xi), r^*_{task1}(Res_0, \xi)\right) \\ &= \frac{r_1 \xi_1}{r_1 \xi_1} \frac{r_3 \xi_4}{r_3 \xi_4} \min\left(r_1 \xi_1, r_3 \xi_4\right) \\ &= \min\left(r_1 \xi_1, r_3 \xi_4\right) \end{aligned}$$

# $f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC



# $f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC




$$r = \frac{r_1}{2r_1}\frac{r_3}{3r_3}\min(2r_1, 3r_3) = \frac{1}{6}\min(2r_1, 3r_3)$$

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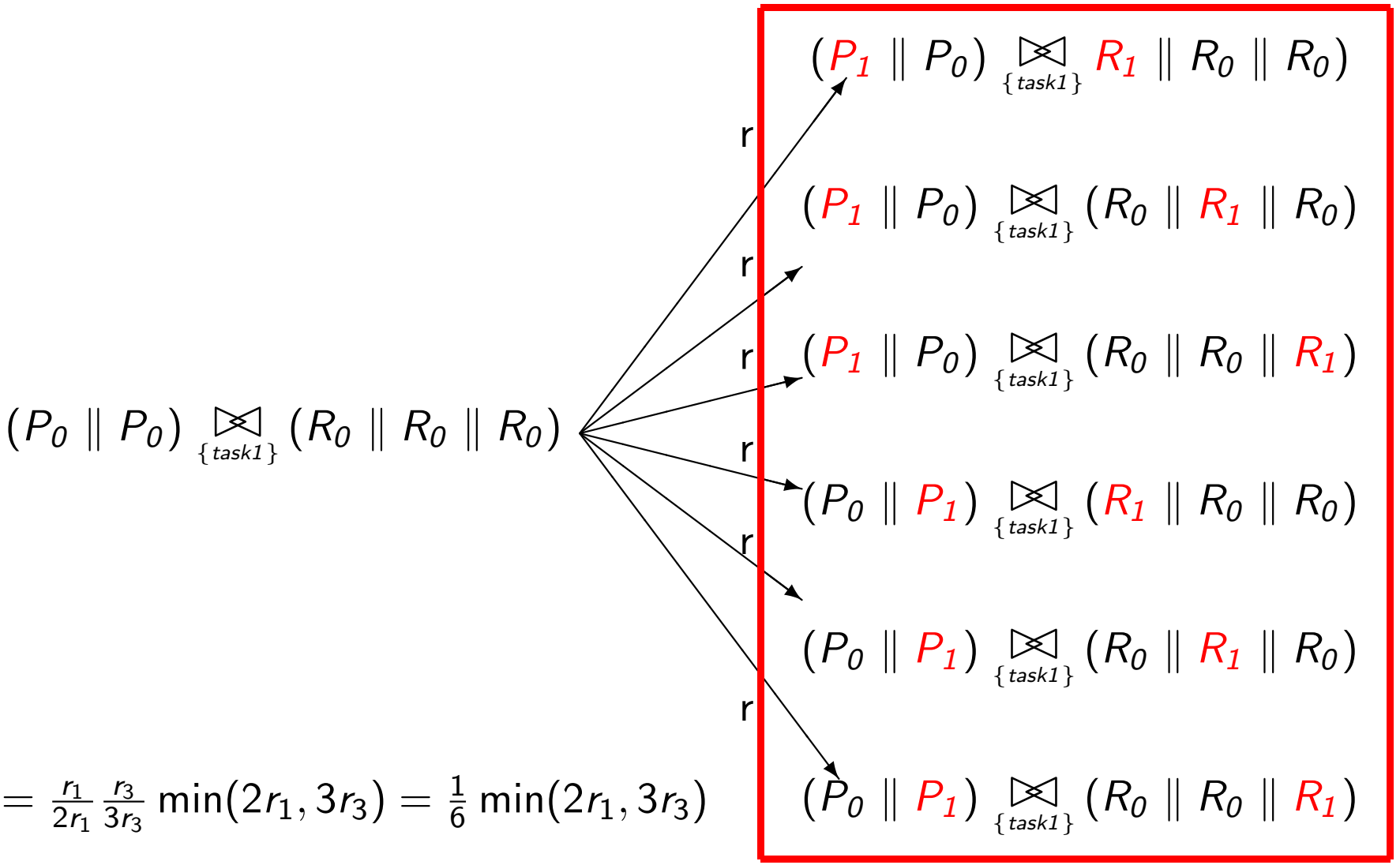


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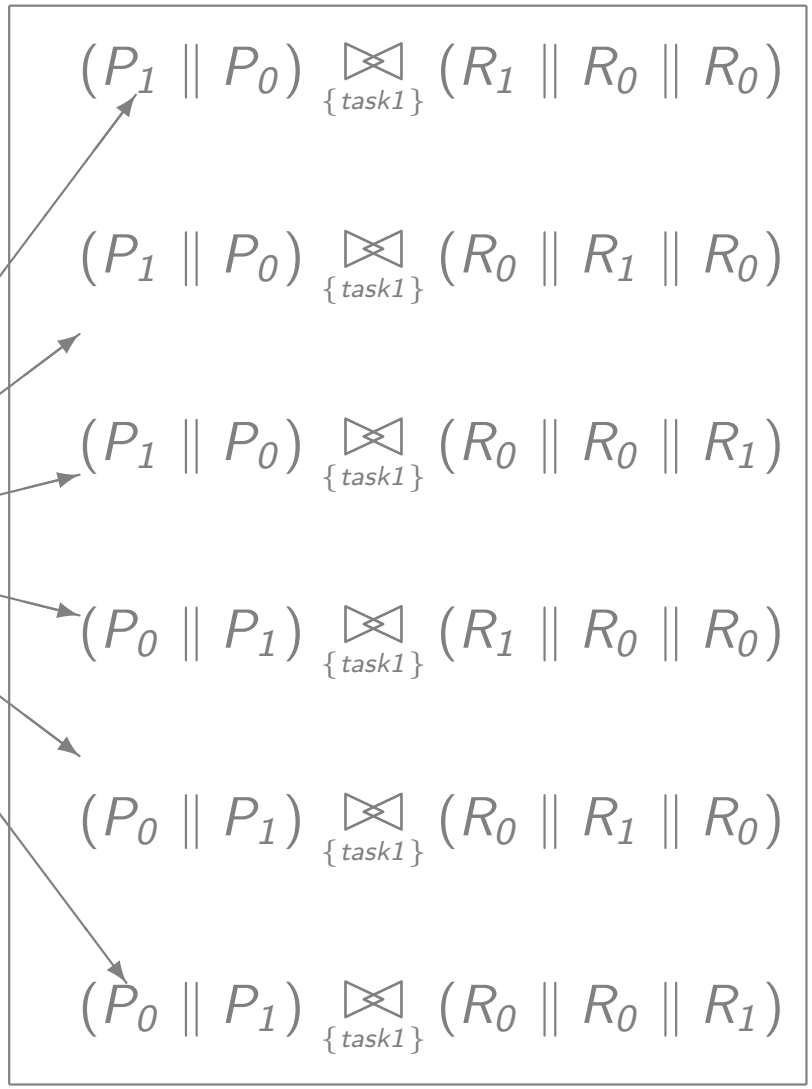
# $f(\xi, l, \alpha)$ as the Generator Matrix of the *Lumped* CTMC

$$(2,0,3,0) \xrightarrow{\min(2r_1, 3r_3)} (1,1,2,1)$$

$$(P_0 \parallel P_0) \underset{\{task1\}}{\bowtie} (R_0 \parallel R_0 \parallel R_0)$$

$$r = \frac{r_1}{2r_1}\frac{r_3}{3r_3}\min(2r_1, 3r_3) = \frac{1}{6}\min(2r_1, 3r_3)$$



# Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

# Jump Multiset

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$r(\xi) = \min\left(r_1\xi_1, r_3\xi_3\right)$$

$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

# Equivalent Transitions

Some transitions may give the same information:

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$
$$Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset,\xi_4 r_4}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_0$$

i.e., $Res_1$ may perform an action independently from the rest of the system.

This is captured by the procedure used for the construction of the generator function $f(\xi, l, \alpha)$

# Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

# Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$

# Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

- Take $l = (0, 0, 0, 0)$
- Add $-1$ to all elements of $l$ corresponding to the indices of the components in the lhs of the transition

$$l = (-1, 0, 0, -1)$$

- Add $+1$ to all elements of $l$ corresponding to the indices of the components in the rhs of the transition

$$l = (-1 + 1, +1, -1) = (0, 0, +1, -1)$$

# Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_1 \xrightarrow{reset, \xi_4 r_4}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

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- Add $+1$ to all elements of $l$ corresponding to the indices of the components in the rhs of the transition

$$l = (-1 + 1, +1, -1) = (0, 0, +1, -1)$$

$$f\left(\xi, (0, 0, +1, -1), reset\right) = \xi_4 r_4$$

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$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_{*} Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$

$$f(\xi, (-1, +1, -1, +1), task1) = r(\xi)$$

# Construction of $f(\xi, l, \alpha)$

$$Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task1, r(\xi)}_* Proc_1 \underset{\{task1\}}{\bowtie} Res_1$$
$$Proc_1 \underset{\{task1\}}{\bowtie} Res_0 \xrightarrow{task2, \xi_2 r_2'}_* Proc_0 \underset{\{task1\}}{\bowtie} Res_0$$

$$\begin{aligned} f(\xi, (-1, +1, -1, +1), task1) &= r(\xi) \\ f(\xi, (+1, -1, 0, 0), task2) &= \xi_2 r_2 \end{aligned}$$

# Construction of $f(\xi, l, \alpha)$

$$
\begin{array}{lll}
Proc_0 \underset{\{task1\}}{\bowtie} Res_0 & \xrightarrow{task1, r(\xi)}_* & Proc_1 \underset{\{task1\}}{\bowtie} Res_1 \\
Proc_1 \underset{\{task1\}}{\bowtie} Res_0 & \xrightarrow{task2, \xi_2 r_2'}_* & Proc_0 \underset{\{task1\}}{\bowtie} Res_0 \\
Proc_0 \underset{\{task1\}}{\bowtie} Res_1 & \xrightarrow{reset, \xi_4 r_4}_* & Proc_0 \underset{\{task1\}}{\bowtie} Res_0
\end{array}
$$

$$
\begin{aligned}
f(\xi, (-1, +1, -1, +1), task1) &= r(\xi) \\
f(\xi, (+1, -1, 0, 0), task2) &= \xi_2 r_2 \\
f(\xi, (0, 0, +1, -1), reset) &= \xi_4 r_4
\end{aligned}
$$

# Capturing behaviour in the Generator Function

$$
\begin{aligned}
Proc_0 &\stackrel{def}{=} (task1, r_1).Proc_1 \\
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## Numerical Vector Form

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{N}^4, \quad \xi_1 + \xi_2 = N_P \text{ and } \xi_3 + \xi_4 = N_R$$

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## Generator Function

$$
f(\xi, l, \alpha): \quad
\begin{aligned}
f(\xi, (-1, 1, -1, 1), task1) &= \min(r_1\xi_1, r_3\xi_3) \\
f(\xi, (1, -1, 0, 0), task2) &= r_2\xi_2 \\
f(\xi, (0, 0, 1, -1), reset) &= r_4\xi_4
\end{aligned}
$$

# Extraction of the ODE from $f$

## Generator Function

$$\begin{aligned} f(\xi, (-1,1,-1,1), \mathit{task1}) &= \min(r_1\xi_1, r_3\xi_3) \\ f(\xi, (1,-1,0,0), \mathit{task2}) &= r_2\xi_2 \\ f(\xi, (0,0,1,-1), \mathit{reset}) &= r_4\xi_4 \end{aligned}$$

## Differential Equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_{\mathcal{M}}(x) = \sum_{l \in \mathbb{Z}^d} l \sum_{\alpha \in \mathcal{A}} f(x, l, \alpha)$$

$$= (-1,1,-1,1)\min(r_1x_1, r_3x_3) + (1,-1,0,0)r_2x_2 + (0,0,1,-1)r_4x_4$$

# Extraction of the ODE from $f$

## Generator Function

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## Differential Equation

$$\begin{aligned} \frac{dx_1}{dt} &= -\min(r_1x_1, r_3x_3) + r_2x_2 \\ \frac{dx_2}{dt} &= \min(r_1x_1, r_3x_3) - r_2x_2 \\ \frac{dx_3}{dt} &= -\min(r_1x_1, r_3x_3) + r_4x_4 \\ \frac{dx_4}{dt} &= \min(r_1x_1, r_3x_3) - r_4x_4 \end{aligned}$$

# Outline

# Internet worms: Background

- Internet worms are malicious programs that exploit operating system security weaknesses to propagate themselves.

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- Worms like Nimbda, Slammer, Code Red, Sasser and Code Red 2 have caused the Internet to become unusable for many hours at a time until security patches could be applied and routers fixed.
- The estimated cost of computer worms and related activities is about $50 billion a year.

# An Internet-scale Problem

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# An Internet-scale Problem

- We wish to study the emergent behaviour of Internet worms as they spread to thousands and then hundreds-of-thousands of hosts.
- Explicit state-based methods for calculating steady-state, transient or passage-time measures are limited to state-spaces of the order of $10^9$.
- By transforming our stochastic process algebra model into a set of ODEs, we can obtain a plot of model behaviour against time for models with global state spaces in excess of $10^{10000}$ states.

# Susceptible-Infective-Removed (SIR) model

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\frac{\mathrm{d}i(t)}{\mathrm{d}t} &= \beta\, s(t)\, i(t) - \gamma\, i(t) \\
\frac{\mathrm{d}r(t)}{\mathrm{d}t} &= \gamma\, i(t)
\end{aligned}$$

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- An infected computer can be patched so that it is no longer infected or susceptible to infection.
- This state is termed removed and is an absorbing state for that component in the system.

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- Additionally, an attempted network connection can fail or timeout as indicated by the *fail* action.
- This might be due to network contention or the lack of availability of a susceptible machine to infect.
- As large scale worm infections tend not to waste time determining whether a given host is already infected or not, we assume that a certain number of infections will attempt to reinfect hosts; in this instance, the host is unaffected.

# Susceptible-Infective-Removed over a network

$$
\begin{aligned}
S &\stackrel{def}{=} (infectS, \top).I \\
I &\stackrel{def}{=} (infectI, \beta).I + (infectS, \top).I + (patch, \gamma).R \\
R &\stackrel{def}{=} Stop
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$$

$$
\begin{aligned}
Net &\stackrel{def}{=} (infectI, \top).Net' \\
Net' &\stackrel{def}{=} (infectS, \beta).Net + (fail, \delta).Net
\end{aligned}
$$

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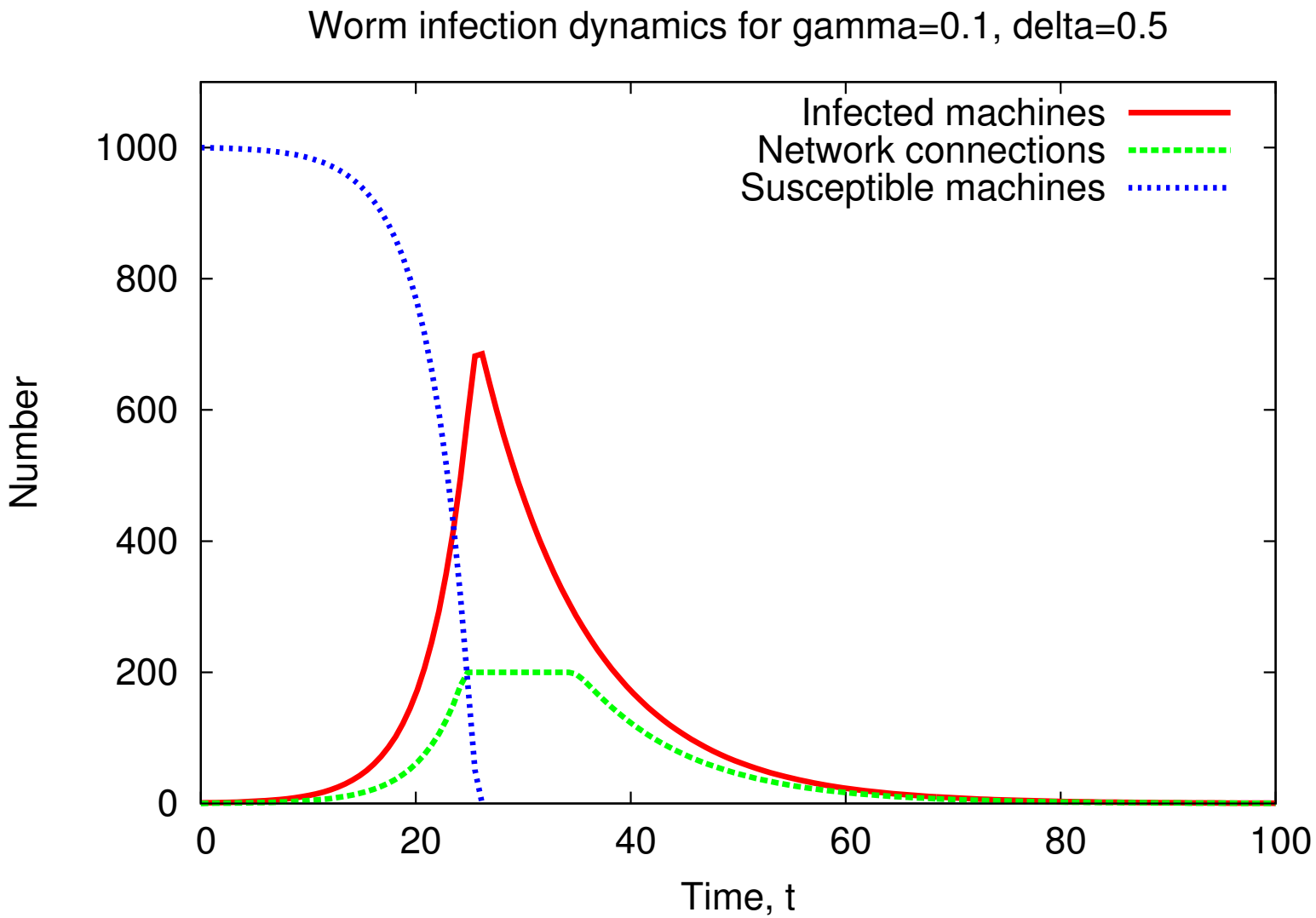
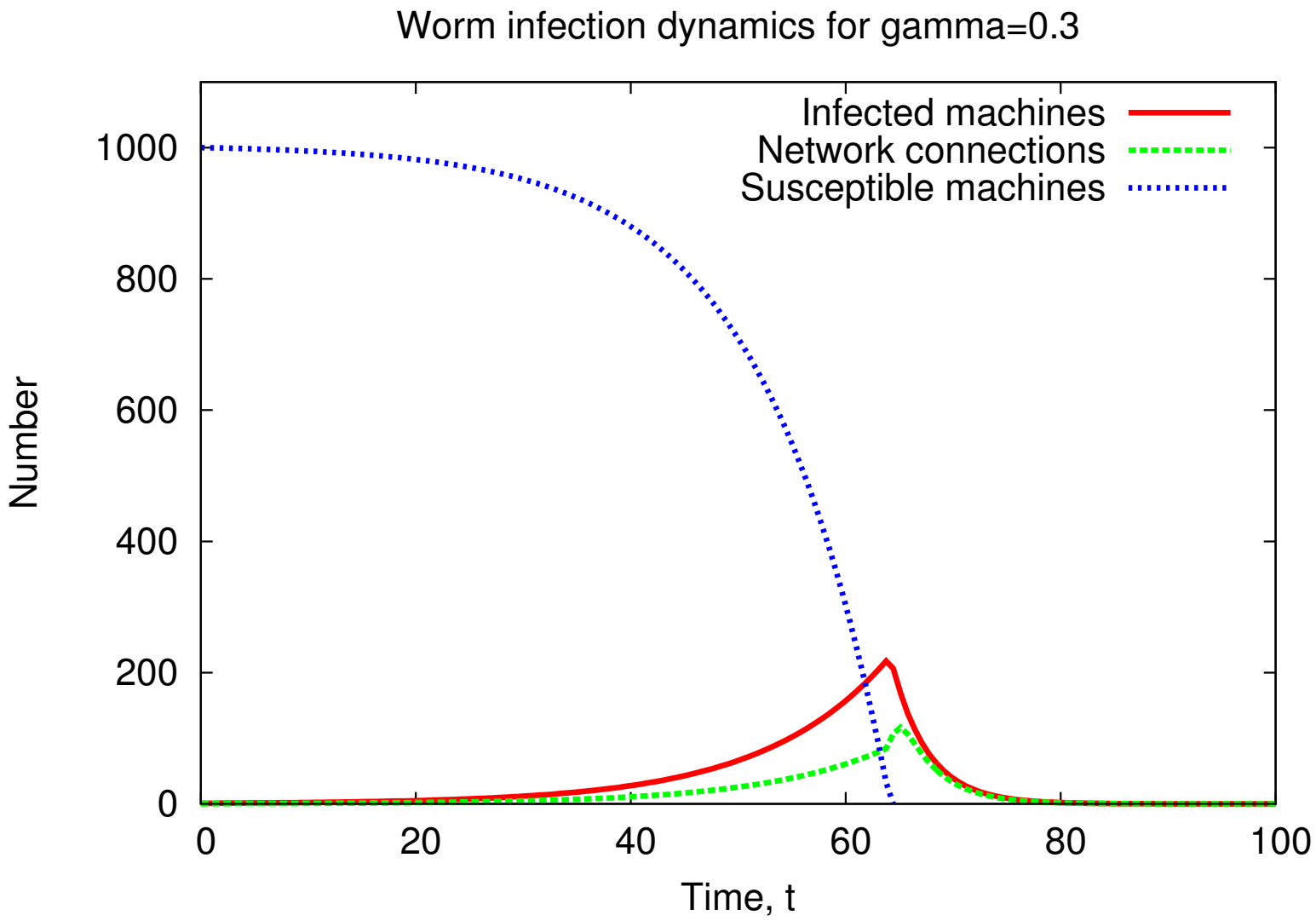
\end{aligned}
$$

$$
Sys \stackrel{def}{=} (S[N] \parallel I) \underset{L}{\bowtie} Net[M] \qquad \text{where } L = \{ infectI, infectS \}
$$

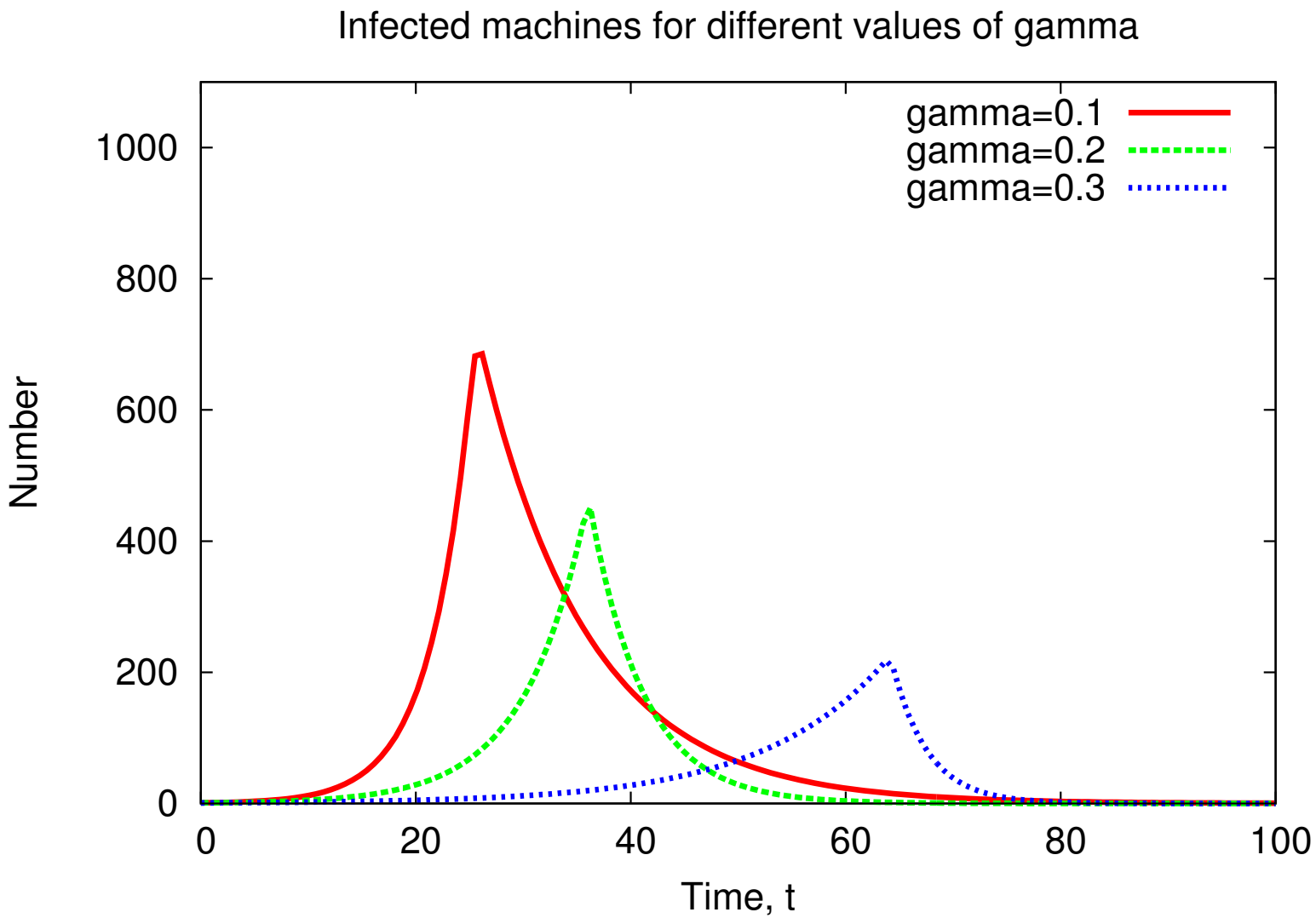
# Patch rate $\gamma = 0.1$. Connection failure rate $\delta = 0.5$



# Patch rate $\gamma = 0.3$. Connection failure rate $\delta = 0.5$



# Increasing machine patch rate $\gamma$ from 0.1 to 0.3



# Susceptible-Infective-Removed-Reinfection (SIRR) model

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- We use this to model a faulty or incomplete security upgrade or the mistaken removal of security patches which had previously defended the machine against attack.

# Susceptible-Infective-Removed-Reinfection (SIRR) model

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\begin{aligned}
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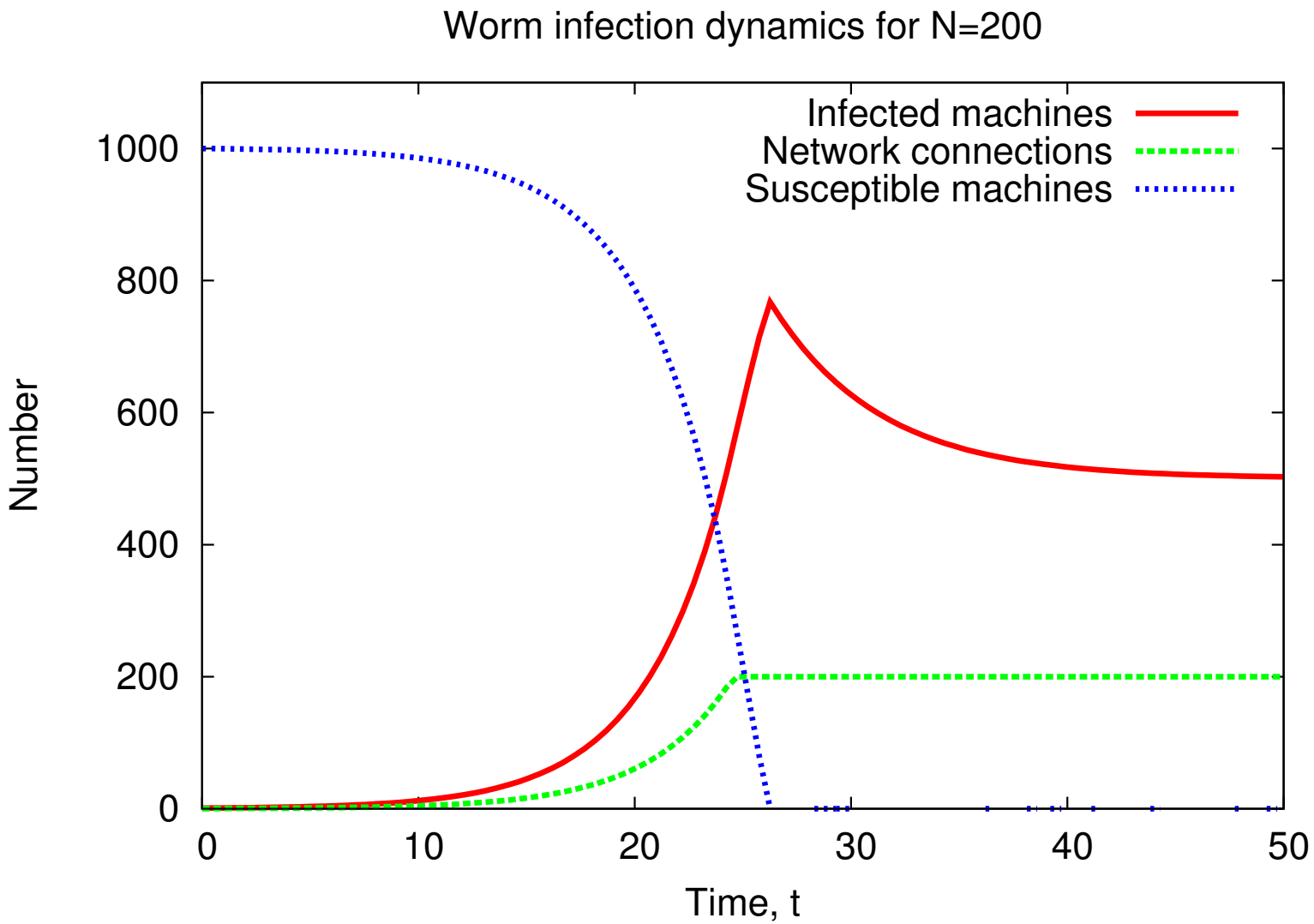
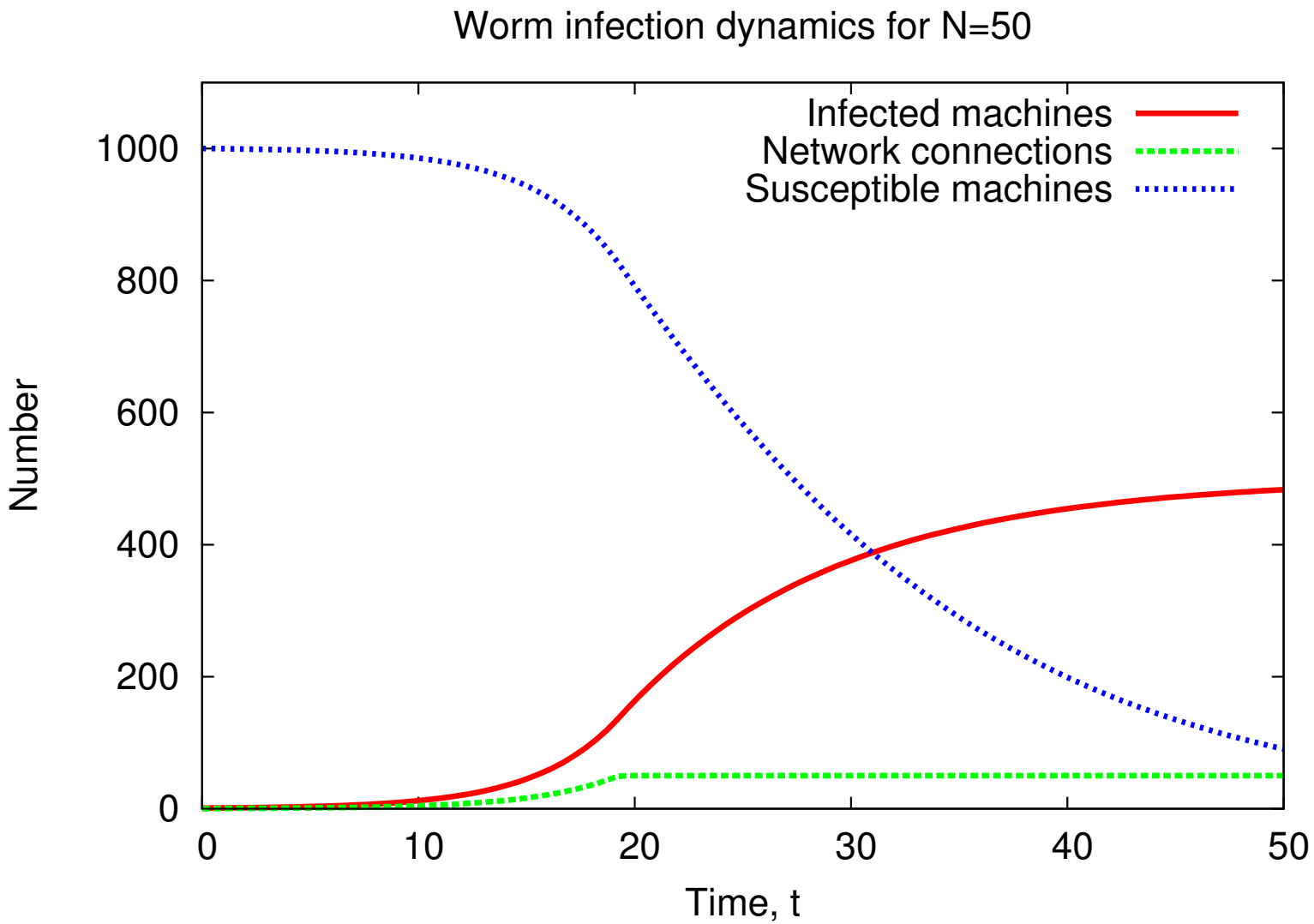
\begin{aligned}
Sys &\stackrel{def}{=} (S[1000] \parallel I) \underset{L}{\bowtie} Net[M] \\
&\qquad\qquad where\ L = \{infectI, infectS\}
\end{aligned}
$$

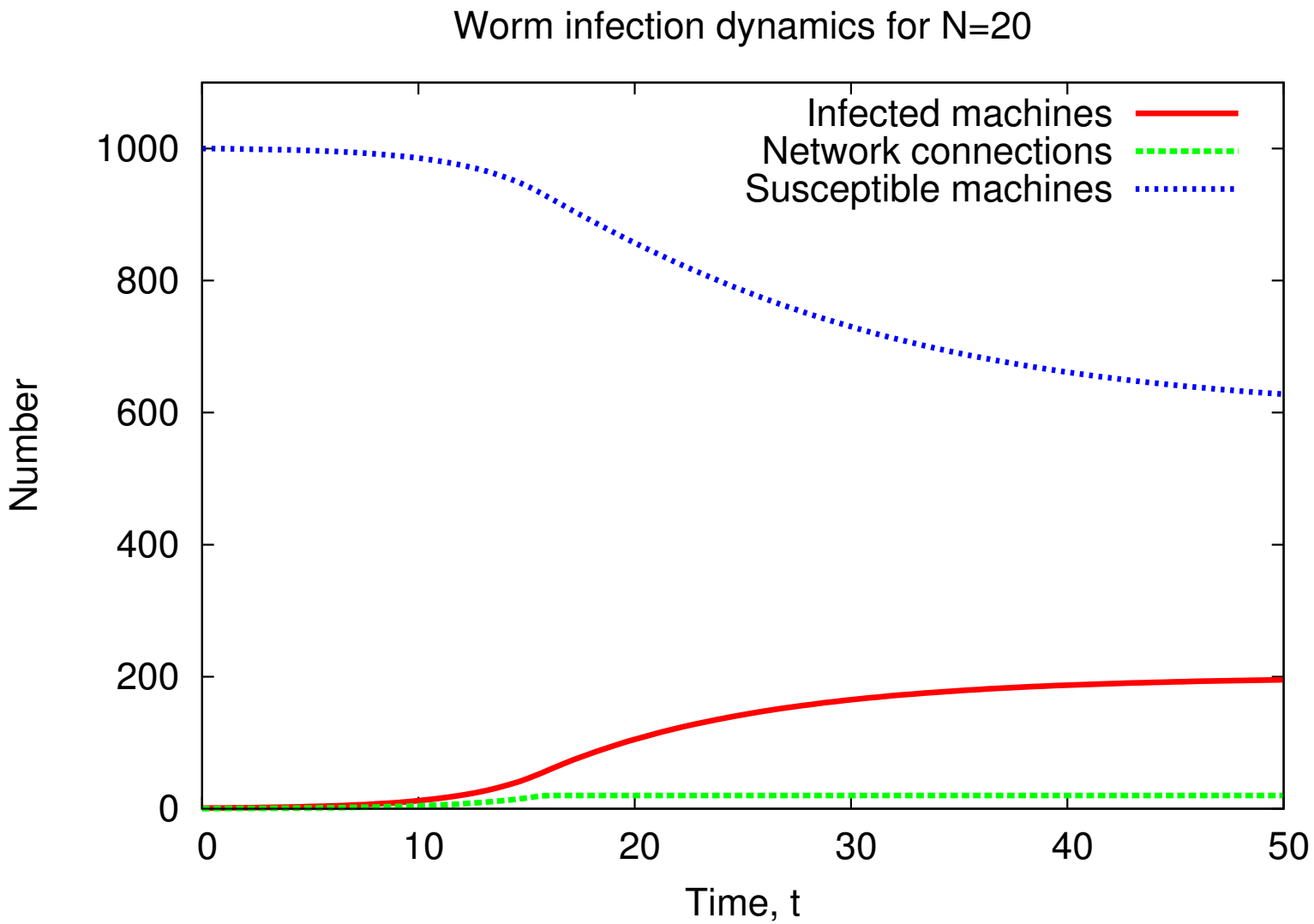
# Unsecured SIR model (200 network channels)



# Unsecured SIR model (50 network channels)



# Unsecured SIR model (20 network channels)



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- The scale of problems which can be modelled in this way vastly exceeds those which are founded on explicit state representations.
- We believe the modelling methods exemplified here to be generally useful for analysing the behaviour of populations of interacting processes with complex dynamics.

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- Services are replicated across the physical sites.
- By agreement in the university, students may connect to any site to download content and use services, not just the one which is geographically closest.

# Case Study: A Virtual University

# Location, Time, and Size

# Replicating Web Services

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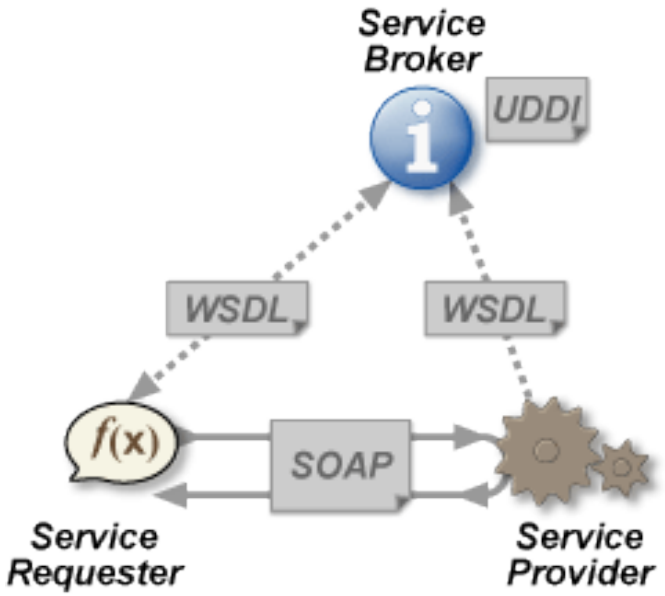

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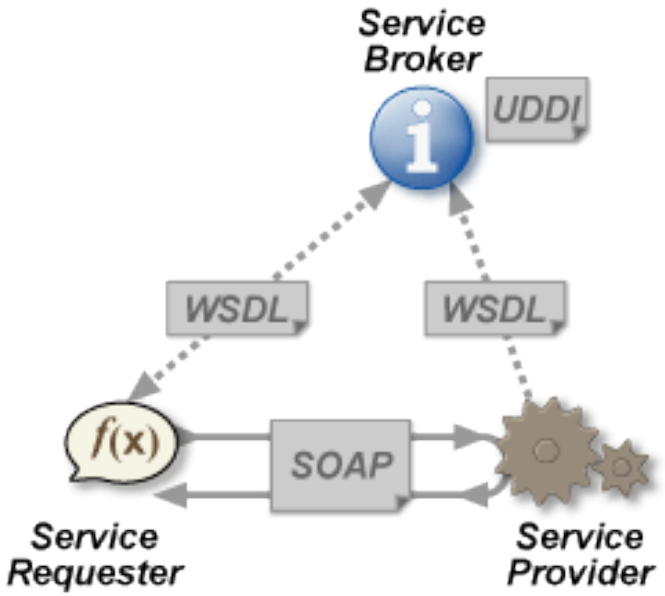


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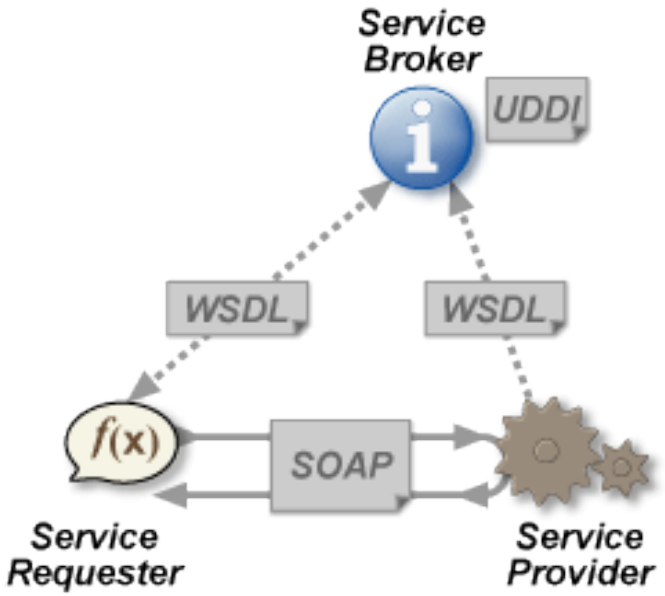


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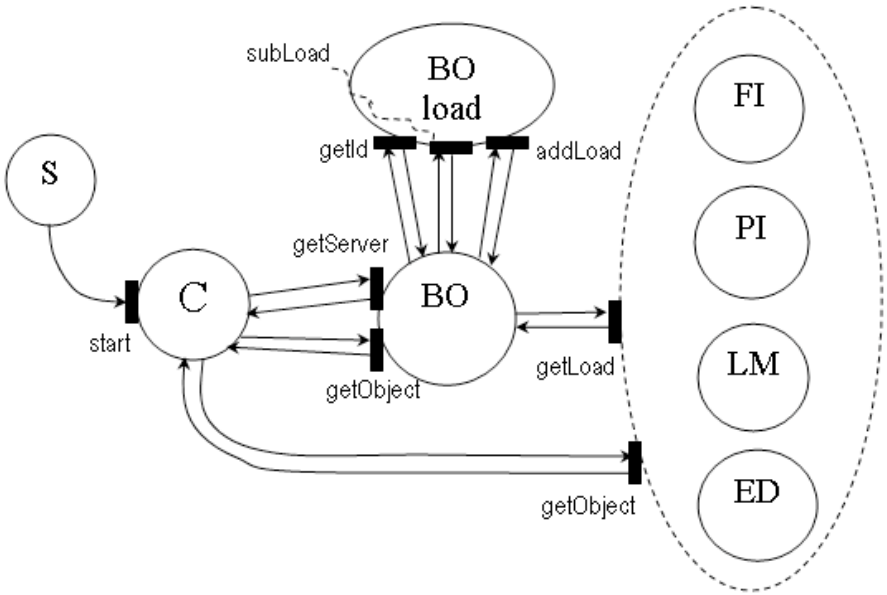


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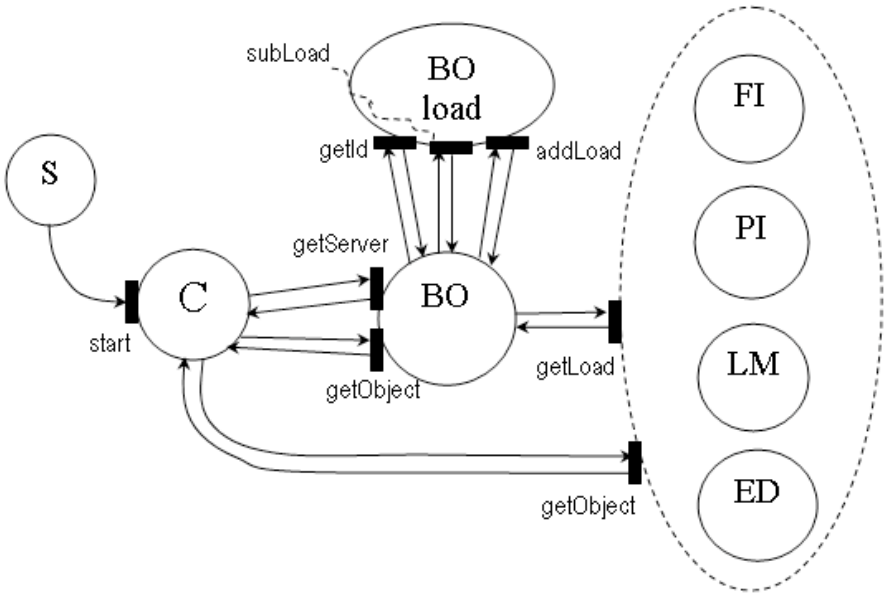


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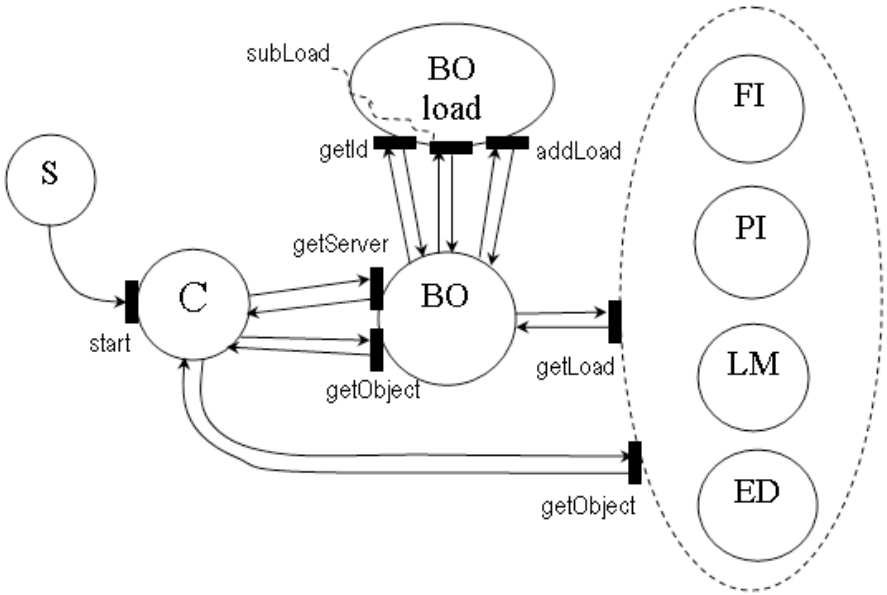


1. A client contacts a university site to download content.
2. The site either serves the request or forwards it to another site.
3. The decision in made in accord with the local service policy.

# Model in PEPA

## Clients

$$
\begin{aligned}
Client_i &\stackrel{def}{=} (connect_1, c_{1,i}).(download_1, d_{1,i}).Idle_i \\
&+ (connect_2, c_{2,i}).(download_2, d_{2,i}).Idle_i \\
&\cdots \\
&+ (connect_m, c_{m,i}).(download_m, d_{m,i}).Idle_i \\
&+ (overload, \top).Client_i \\
Idle_i &\stackrel{def}{=} (idle, r_{idle,i}).Client_i
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$$(1 \leq i \leq k)$$

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# Model in PEPA

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$$\begin{aligned} Mirror_j &\stackrel{def}{=} \left(connect_j, f_j(s)\right).MirrorUploading_j \\ MirrorUploading_j &\stackrel{def}{=} \left(download_j, \top\right).Mirror_j \end{aligned}$$

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# Service policies as functional rates in PEPA

## The Bologna policy

Serve all requests while load is less than 75%. If more, and the loads at UNIFI, UPISA, LMU and UEDIN are at least 60%, 60%, 40% and 20% then serve the request if load is less than 95%.

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$$f_{\mathrm{UNIBO}} = \begin{cases} \top & \text{if } \mathit{MirrorUploading}_{\mathrm{UNIBO}} < 75 \\ \top & \text{if } \mathit{MirrorUploading}_{\mathrm{UNIBO}} < 95, \\ & \quad \mathit{MirrorUploading}_{\mathrm{UNIFI}} \geq 60, \\ & \quad\quad \mathit{MirrorUploading}_{\mathrm{UPISA}} \geq 60, \\ & \quad\quad\quad \mathit{MirrorUploading}_{\mathrm{LMU}} \geq 40, \\ & \quad\quad\quad\quad \mathit{MirrorUploading}_{\mathrm{UEDIN}} \geq 20 \\ 0 & \text{otherwise} \end{cases}$$

# Model in PEPA

## Dealing with overload

$$Overload \stackrel{def}{=} \big(overload, o(s)\big).Overload$$

$$o(s) = \begin{cases} \top & f_i(s) = 0, \quad 1 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

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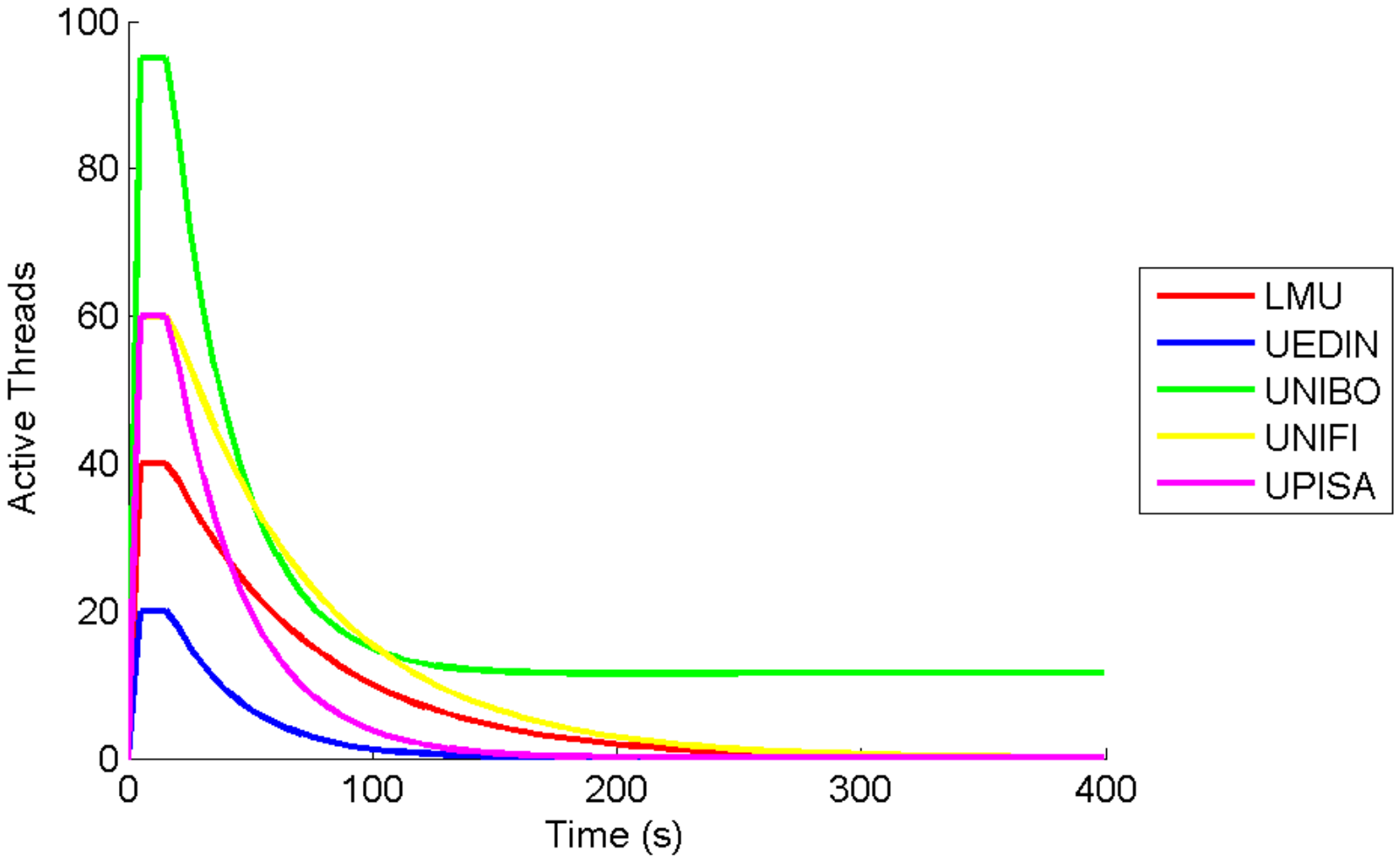
# Numerical Results

$$r_{idle} = 0.001$$



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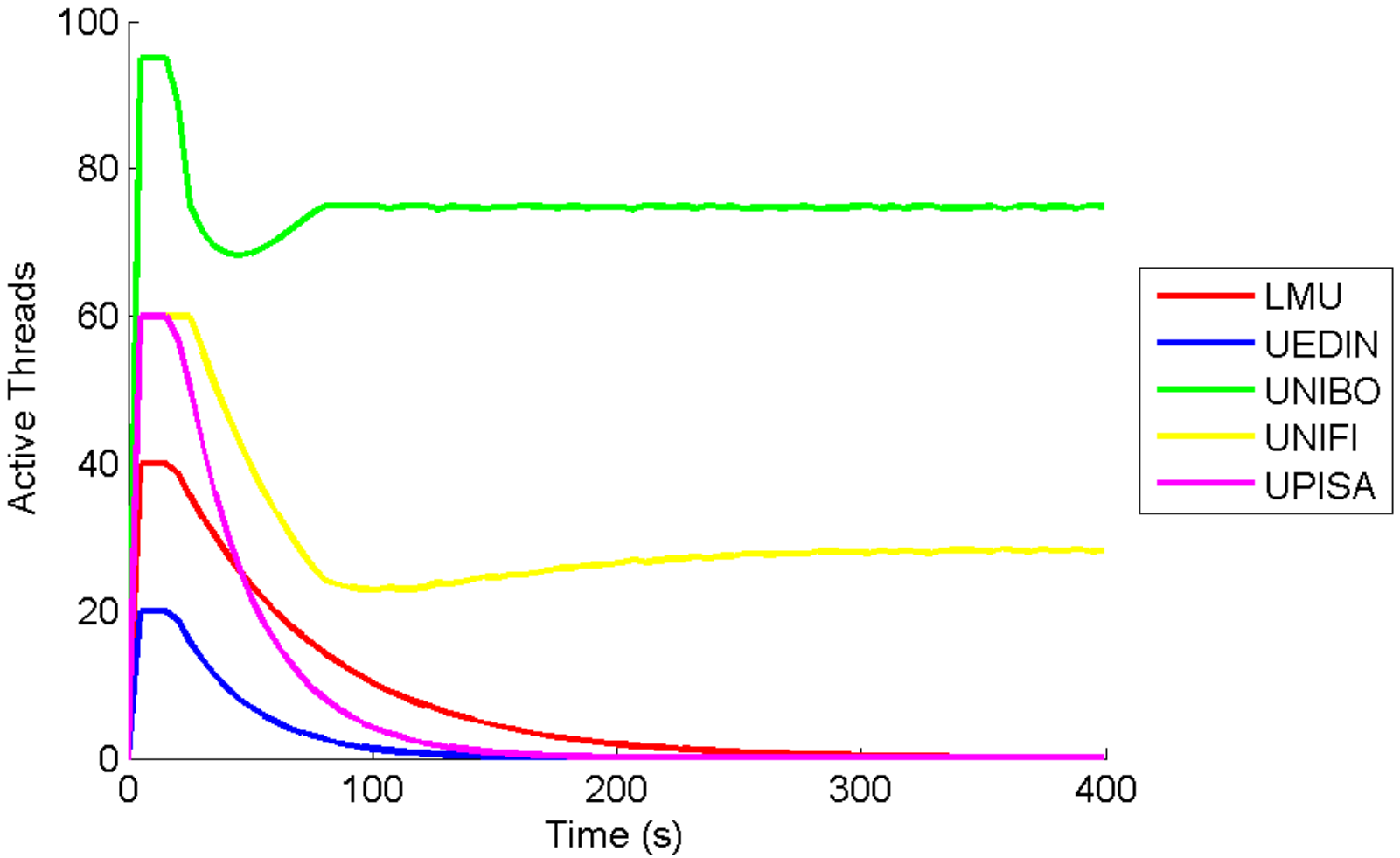

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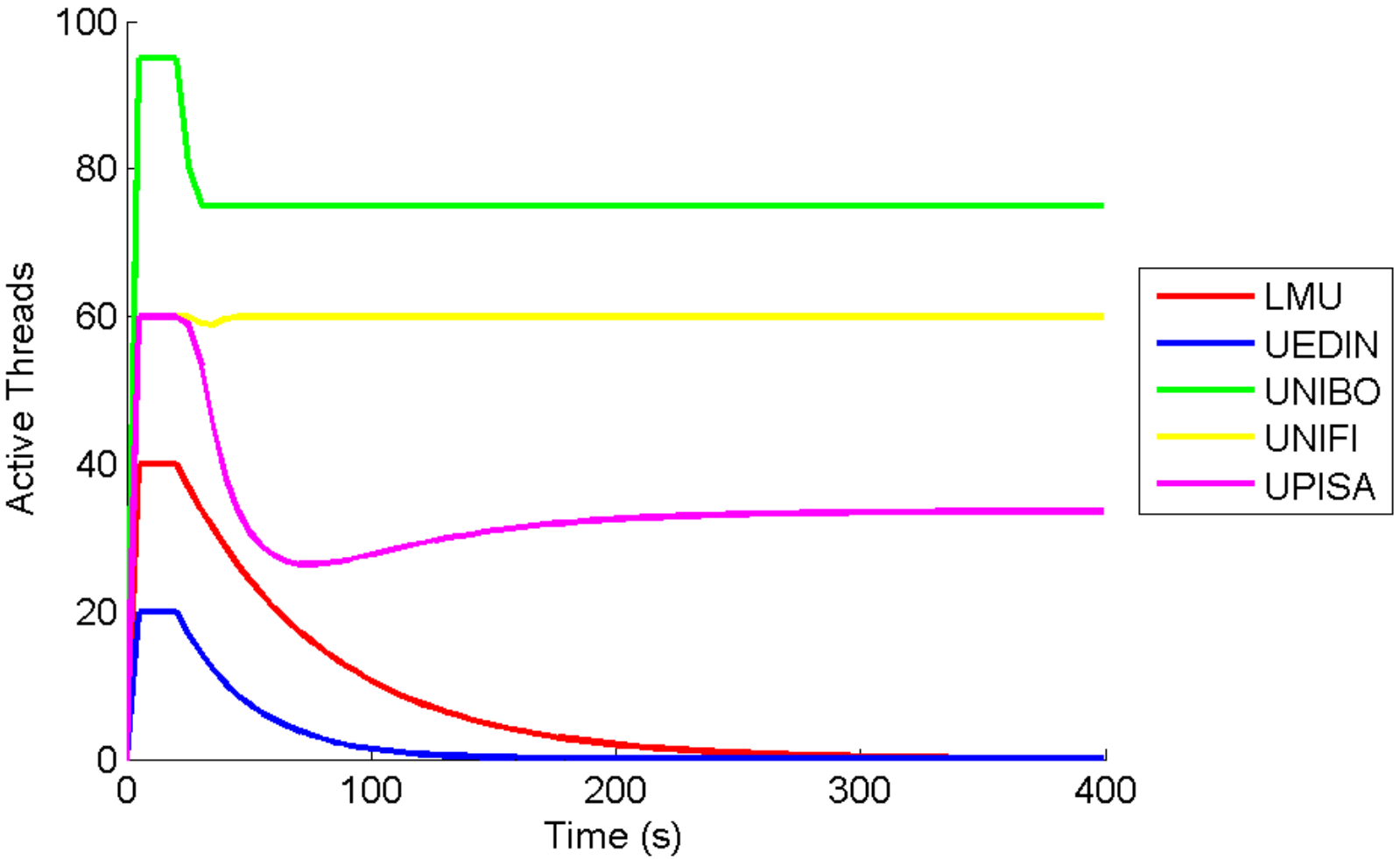

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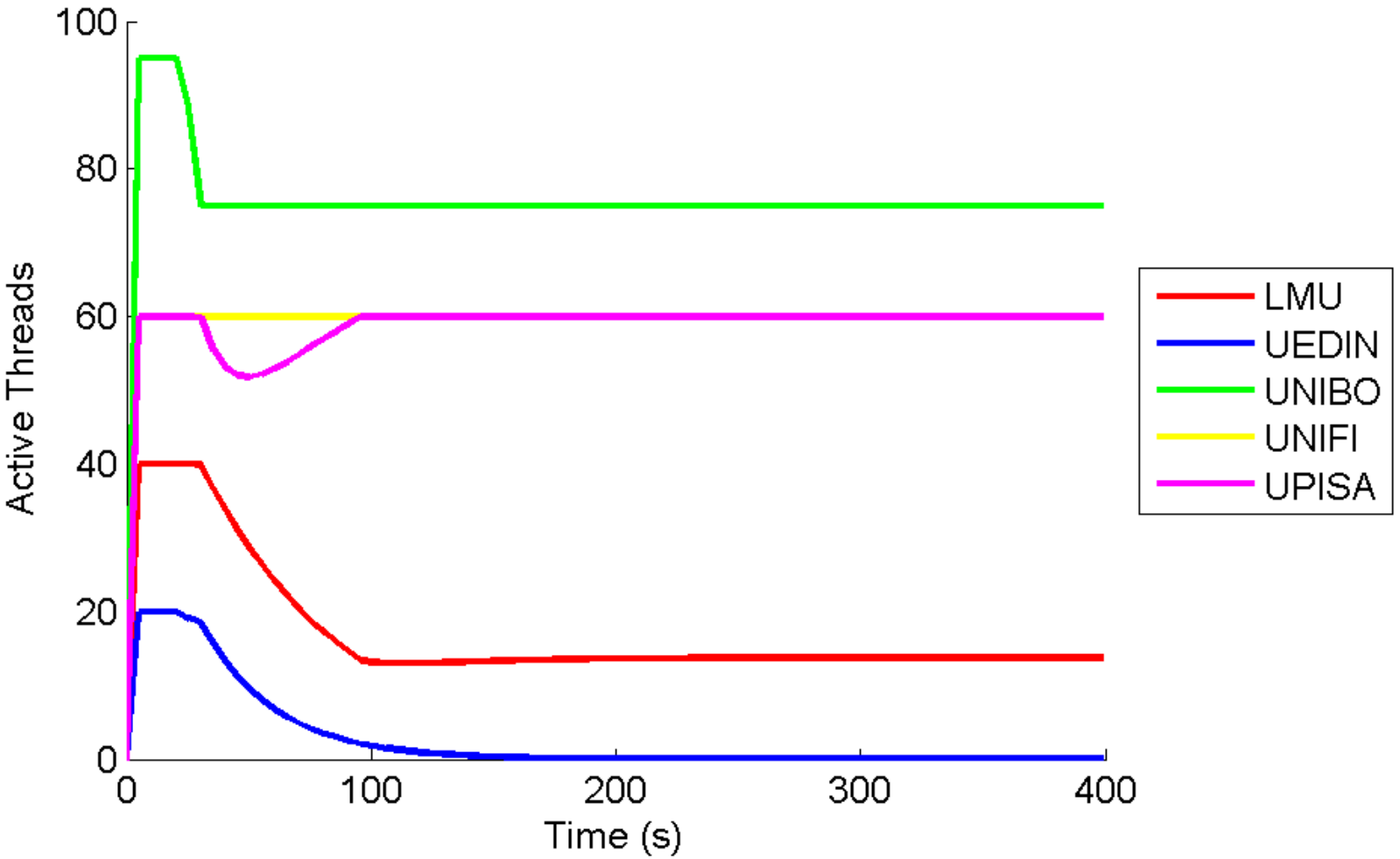

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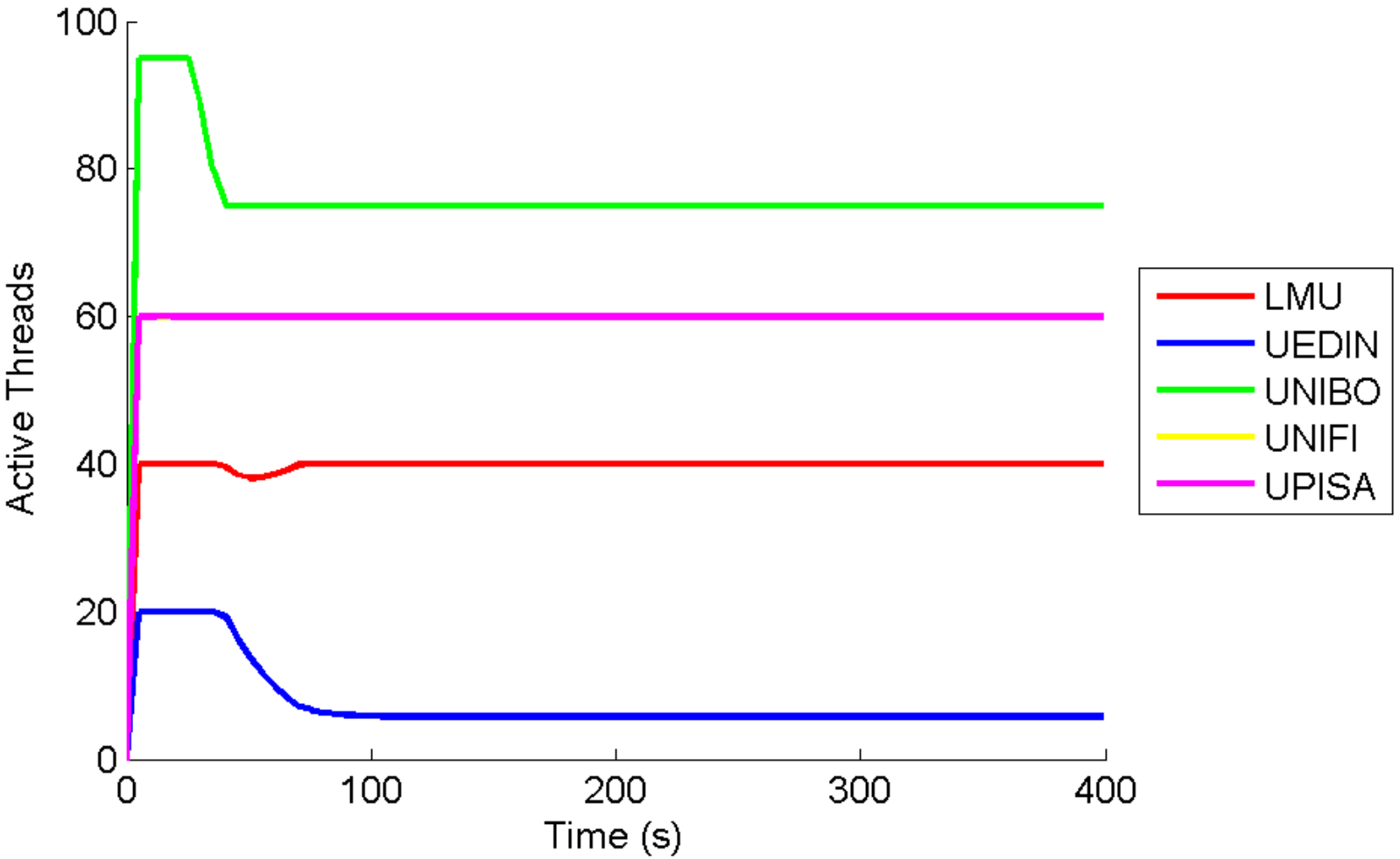

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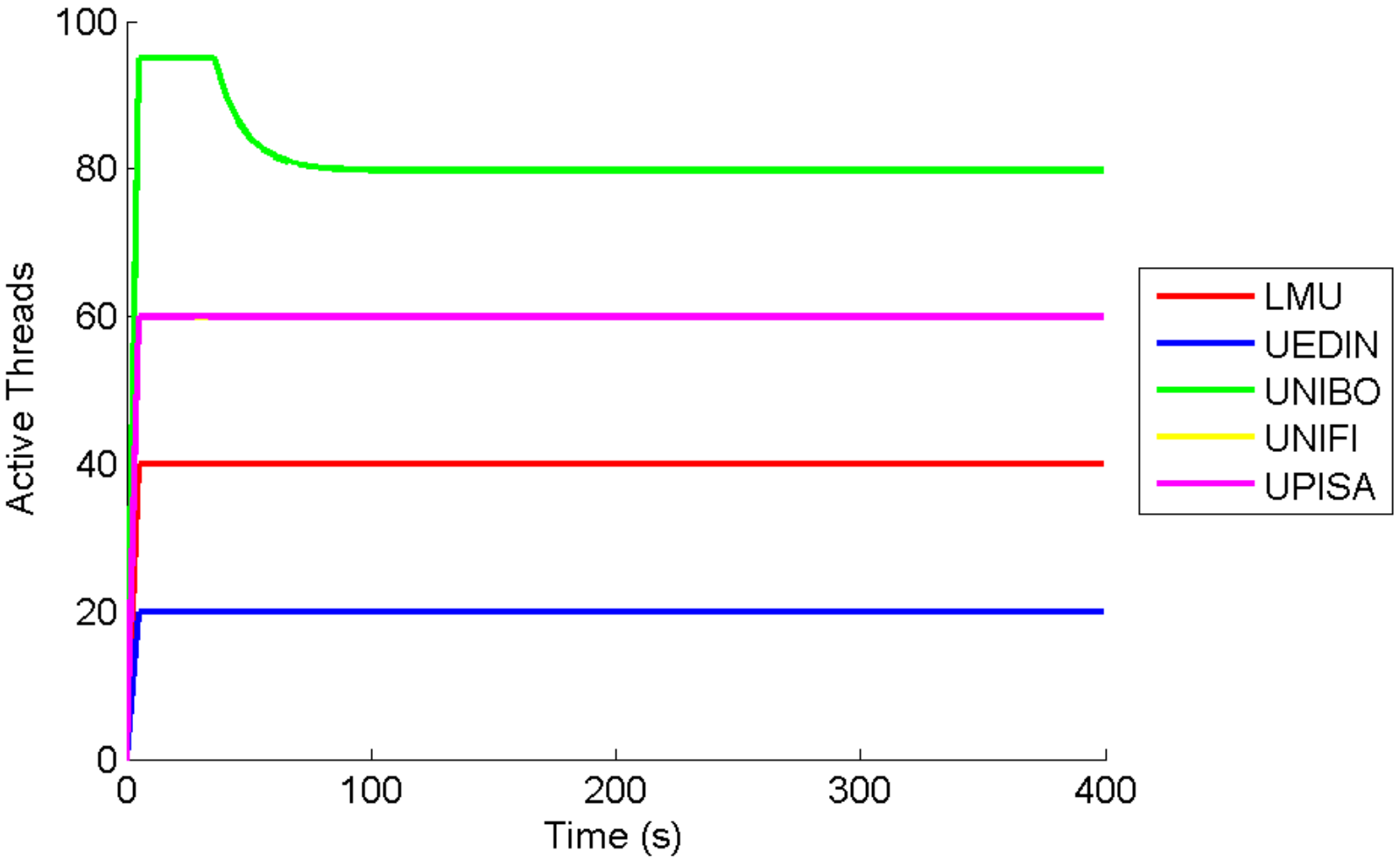

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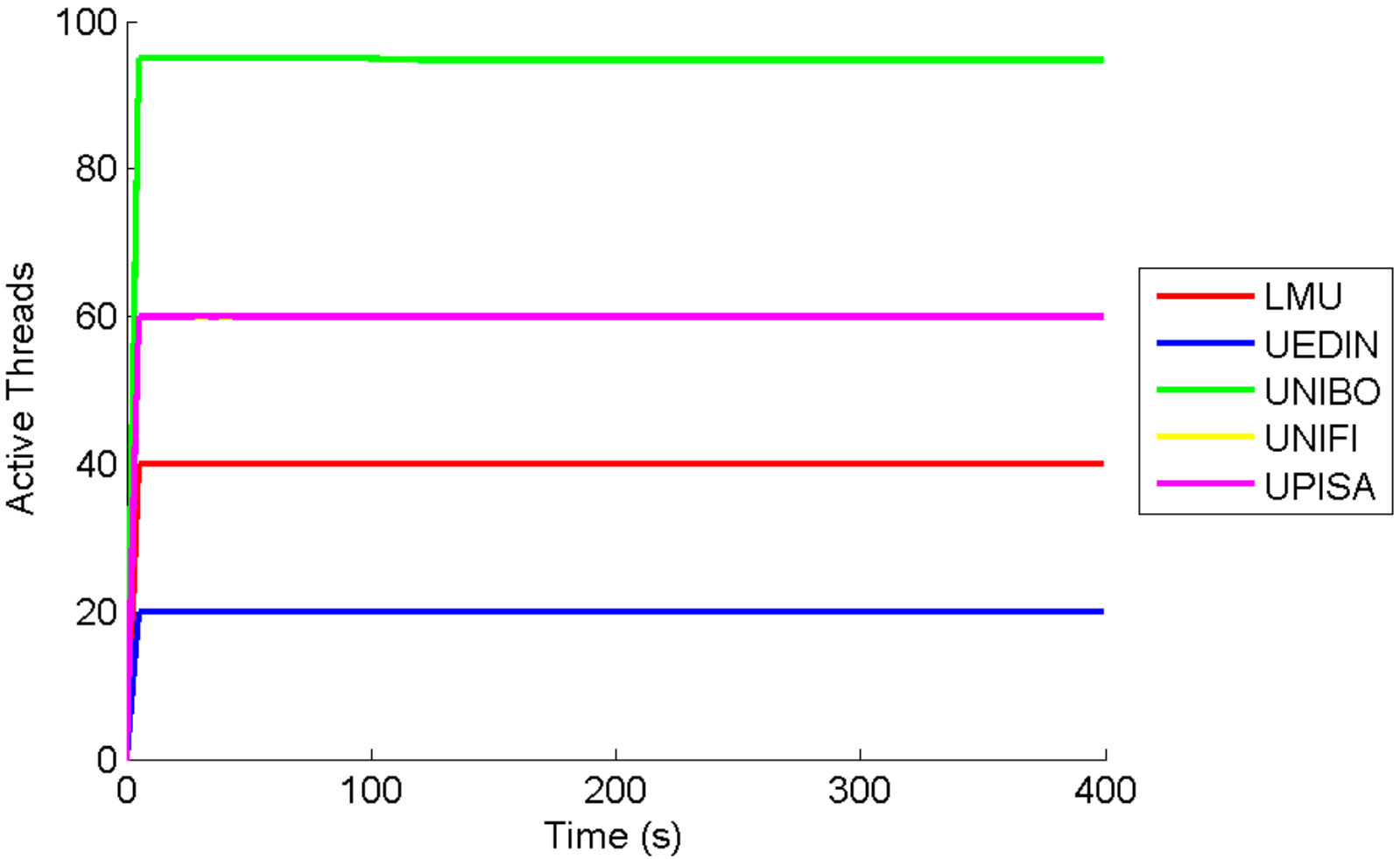

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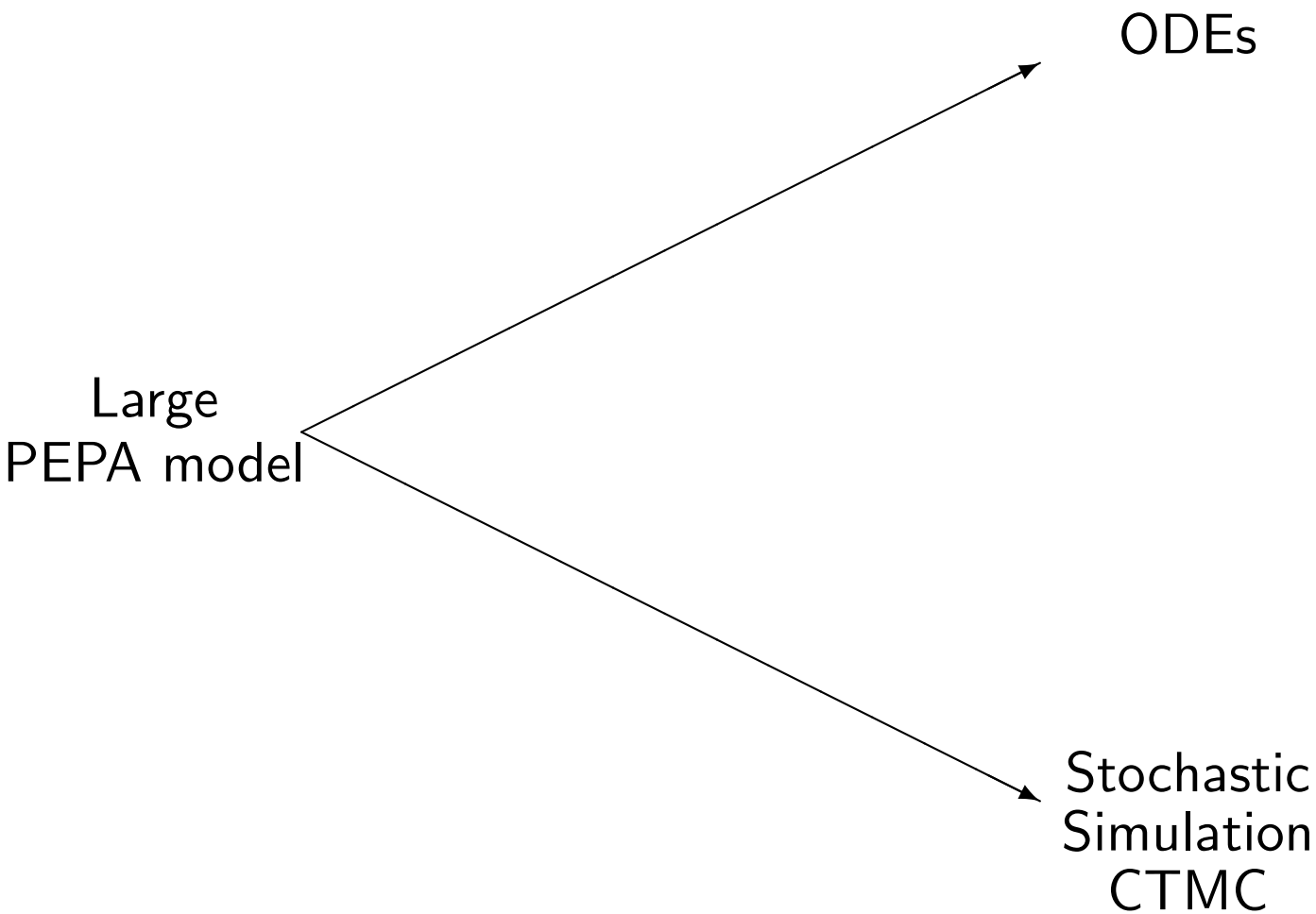
# Outline

# Alternative Representations



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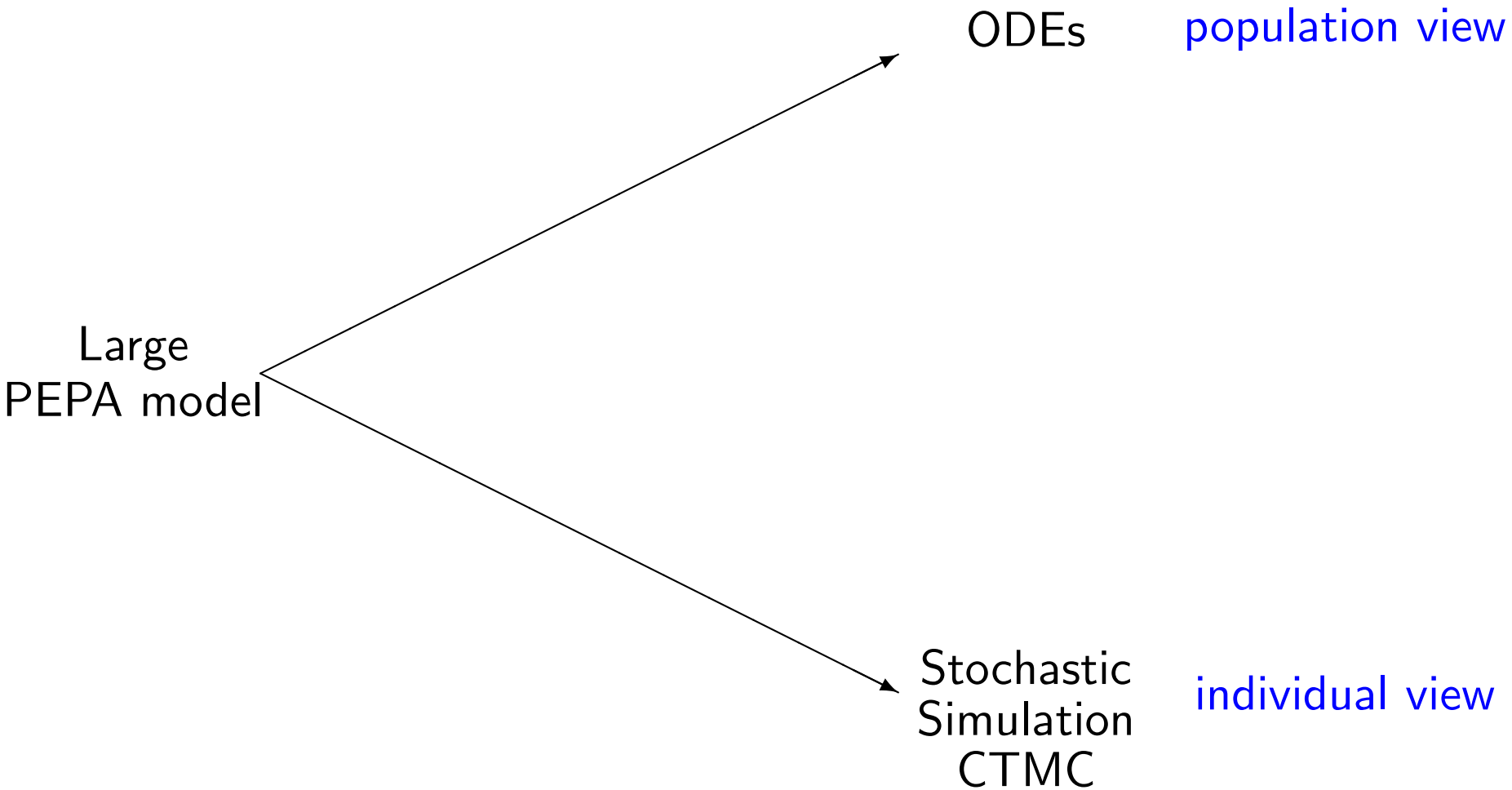

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# Conclusions

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- This alternative view of systems has opened up many and exciting new research directions.

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**More information:**

http://www.dcs.ed.ac.uk/pepa