On the Interpretation of Intuitionistic Logic

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The present article may be considered from two quite different standpoints.

1. If one does not acknowledge the epistemological assumptions of Intuitionism, then only one should consider only the first Section, whose results can be summarised roughly as follows:

   Alongside theoretical logic, which systematises the proof schemata of theoretical truths, one can systematise the schemata of problem-solving, *e.g.* those of geometric construction problems. Then, corresponding to the principle of the syllogism, we find for example, the following principle: *If we can reduce the solution of b to that of a, and reduce the solution of c to that of b, then we can also reduce the solution of c to that of a.*

   One can introduce a corresponding symbolic language, and give the formal calculation rules for the symbolic structure of the system of such problem-solving schemata. One thereby obtains, alongside theoretical logic, a new *calculus of problems.* For that one needs no special epistemological, for example intuitionistic, commitments.

   Then the following remarkable fact holds: *The calculus of problems coincides with Brouwer's intuitionistic logic, as recently formalised by Heyting*[1].

2. In the second Section, intuitionistic logic is examined critically, under the general assumptions of Intuitionism; it is shown there that it should be replaced by the calculus of problems, whose objects are, in reality, not theoretical statements, but rather problems.

*Translated from the German by James McKinna, Edinburgh 2014

[1]Heyting, The formal rules of intuitionistic logic, Proceedings of the Prussian Academy of Science, I, p. 42, II, p. 57, III, p.158, 1930.

§1.

We will not define what a *problem* is, but will explain this concept through a few examples. These are [examples of] problems:

1. To find four integers x, y, z, n for which the relations

   (1) $$x^n + y^n = z^n, \qquad n > 2$$

   hold.

2. To prove the falsity of the Fermat proposition.

3. To draw a circle through three given points (x, y, z)[2]

4. Provided that one root of the equation $ax^2 + bx + c = 0$ is given, to find the other.

5. Provided that the number π can be expressed as a ratio:

   $$\pi = \frac{n}{m}$$

   to find an analogous expression for the number e.

That the second problem is different from the first is clear, and makes no special intuitionistic claim[3]. The fourth and fifth problems are examples of *conventional* problems; while the presupposition of the fifth problem is impossible, and as a consequence the problem is itself *content-free*. The proof, that a problem is content-free, is further always to be regarded as its solution.

We believe, that after these examples and explanations, the concepts "problem" and "solution of the problem" can be used without misunderstanding[4] in all cases which arise in concrete domains of mathematics. Henceforth problems will be indicated with small Latin letters $a, b, c, \ldots$.

If a and b are two problems, $a \wedge b$ denotes the problem "solve both problems a and b", while $a \vee b$ denotes the problem "solve at least one of the problems a and b". Further, $a \supset b$ is the problem "given the solution to a, solve b", or, which means the same, "reduce to the solution of b to the solution of a".

Earlier we did not prescribe that every problem has a solution. If for example the Fermat proposition is correct, then the solution to the first problem would be contradictory. Correspondingly, $\neg a$ denotes the problem "given a solution to a, obtain a contradiction"[5].

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[2] To be completely precise: one should of course in formulating this problem admit the allowable means of construction.

[3] The *statements* "the Fermat proposition is false" and "four integers exist satisfying (1)" are by contrast equivalent from the standpoint of classical logic.

[4] The principal concepts of sentential logic, "statement" and "proof of a statement" stand on no better footing.

[5] It should be observed that one is not to understand the problem $\neg a$ as "to prove the unsolvability of a". If one in general regards "the unsolvability of a" as a well-defined concept, then one obtain only the proposition that from $\neg a$ follows the the unsolvability of a, but not the converse. If it were for example to be proved, that carrying out a well-ordering of the continuum were beyond our capabilities, one could not assert that from the existence of such a well-ordering a contradiction would follow.

After these definitions, let each formula $p(a, b, c, \ldots)$ formed from problems $a, b, c, d, \ldots$ with the help of the signs $\wedge, \vee, \supset, \neg$ also denote a problem. If however $a, b, c, \ldots$ are symbols standing for indeterminate problems, then one says that $p(a, b, c, \ldots)$ is a function of the problem variables $a, b, c, \ldots$. In general, one intends by $(x)\ a(x)$ the problem "to provide a general method of solving $a(x)$ for each particular value of x", if x is a variable (of whatever sort) and $a(x)$ denotes a problem whose meaning depends on the value of x. One should understand this as follows: to solve the problem $(x)\ a(x)$ means to be capable, for every given individual value x_0 of x, to solve the problem $a(x_0)$ after a finite sequence of known steps (determined in advance, even before any choice of x_0)[6].

For functions $p(a, b, c, \ldots)$ in the indeterminate problems $a, b, c, \ldots$ one further writes, instead of

$$(a)(b)(c) \ldots p(a, b, c, \ldots)$$

simply

$$\vdash p(a, b, c, \ldots).^{6a}$$

Thus $\vdash p(a, b, c, \ldots)^{\dagger}$ denotes the problem "to provide a general method of solving $p(a, b, c, \ldots)$ for each particular choice of problems $a, b, c, \ldots$".

Problems of the form $\vdash p(a, b, c, \ldots)$, where p is expressed in terms of the signs $\wedge, \vee, \supset, \neg$ are then the subject of the *elementary problem calculus*[7].

The corresponding functions $p(a, b, c, \ldots)$ are the *elementary problem functions.*

Now, that *I* may have solved a problem, is a purely subjective matter, which in itself is of no interest. But logical and mathematical problems possess a special characteristic, that of *the general validity of their solution*: if I have solved a logical or mathematical problem, then I can present this solution so that is is generally understood, and it is *necessary* that it be recognised as a correct solution, even if this necessity is of a rather ideal character, since it demands a certain intelligence on the part of the audience[8].

The actual purpose of the problem calculus lies in giving, by the mechanistic application of some simple rules of calculation, a method for solving problems of the form $\vdash p(a, b, c, \ldots)$, where $p(a, b, c, \ldots)$ is an elementary problem function. In order to be able to reduce everything to these rules of calculation, however, we must assume that the solutions to some elementary problems are already known. We take as *postulates* that we already have solutions to the following

[6] Here, as earlier, we hope that this definition can lead to no misunderstanding in concrete domains of mathematics.

[6a] This explanation of the meaning of the sign $\vdash$ is quite different from that of Heyting, even though it leads to the same rules of the calculus.

[†] Translator's note. The original has "- $p(a, b, c, \ldots)$", but this is likely a printing error.

[7] This definition is the analogue of the elementary sentential calculus. However, in the sentential calculus, the analogous logical functions $\wedge, \vee, \supset, \neg$may be expressed using only two among them. In the problem calculus, all four functions are independent.

[8] All of this also holds word for word for the proofs of theoretical statements. Nevertheless it is important that each proved statement be *correct*; but for problems one has no such corresponding notion of correctness.

groups A and B of problems. The subsequent presentation is directed only to the reader who has solved all these problems[9].

$$A \begin{cases} 2.1. & \vdash . a \supset a \wedge a .^{10} \\ 2.11. & \vdash . a \wedge b \supset b \wedge a . \\ 2.12. & \vdash . a \supset b . \supset . a \wedge c \supset b \wedge c . \\ 2.13. & \vdash . a \supset b . \wedge . b \supset c . \supset . a \supset c . \\ 2.14. & \vdash . b \supset . a \supset b . \\ 2.15. & \vdash . a \wedge . a \supset b . \supset b . \\ 3.1. & \vdash . a \supset a \vee b . \\ 3.11. & \vdash . a \vee b \supset b \vee a . \\ 3.12. & \vdash . a \supset c . \wedge . b \supset c . \supset . a \vee b \supset c . \\ 4.1. & \vdash . \neg a \supset . a \supset b . \\ 4.11. & \vdash . a \supset b . \wedge . a \supset \neg b . \supset \neg a \end{cases}$$

We assume that the reader can solve those problems which follow the $\vdash$ sign for every choice of problems a, b, c. That presents no difficulty. In problem (2.12) for example, on the assumption that the solution of a has been reduced to that of b, one has to reduce the solution of $b \wedge c$ to that of $a \wedge c$. Suppose given a solution to $a \wedge c$; that means we have given, as well as a solution to a, a solution to c; by hypothesis, we derive a solution to b from the solution to a; since the solution to c is given, we obtain solutions to both problems b and c, and thus to $a \wedge c$. From such considerations we obtain a general method for the solution to the problem

$$a \supset b . \supset . a \wedge c \supset b \wedge c$$

valid for whatever a, b, c. We thus have the right to regard the problem

$$2.13. \vdash . a \supset b . \supset . a \wedge c \supset b \wedge c .$$

(with the generalisation sign $\vdash$) as solved.

In particular what problem 4.1 entails is that, as soon as $\neg a$ is solved, then it is impossible to solve a, and thus the problem $a \supset b$ is content-free.

The second Group B of problems, for which we postulate that solutions may be given, contains only three problems[11]. Namely we assume that we are in a position (*i.e.* we possess a general method) to solve the following problems, for any given elementary problem functions $p, q, r, s, \ldots$:

I. If $\vdash p \wedge q$ is solved, to solve $\vdash p$

II. If $\vdash p$ and $\vdash p \supset q$ are solved, to solve $\vdash q$

[9] In the case of the sentential calculus, if one wishes to confirm the correctness of its consequences, one should first convince oneself of the correctness of the axioms.

[10] For the numbering of the formulas and the use of the separators (.), cf. Heyting I.

[11] They cannot, however, be expressed using the symbols of the elementary problem calculus.

III. If $\vdash p(a, b, c, \ldots)$ is solved, to solve $\vdash p(q, r, s, \ldots)$

Now we can give the *rules* of our calculus of problems:

1. Firstly, we place all the problems of Group A on the list of solved problems.
2. If $\vdash p \wedge q$ is already on our list, then we are allowed to add $\vdash p$ to it.
3. If $\vdash p$ and $\vdash p \supset q$ are already there, then we are further allowed to add $\vdash q$.
4. If $\vdash p(a, b, c, \ldots)$ is already on our list, then we may add $\vdash p(q, r, s, \ldots)$.

One may easily convince oneself on the basis of the earlier postulates that these formal rules really guarantee solutions to the corresponding problems.

We dispense with further pursuit of such calculations, for all the foregoing rules of the calculus and *a priori* formulas we have written coincide with the axioms and rules of Heyting's first article[12]; in what follows we can interpret all the formulas of that article as problems, and take these problems as having been solved.

We merely note here some especially interesting problems among those (and thus should also regard them as having been solved):

$$
\begin{aligned}
&4.3.\ \vdash .\, a \supset \neg\neg a\, . \\
&4.2.\ \vdash .\, a \supset b\, . \supset .\, \neg b \supset \neg a\, . \\
&4.32.\, \vdash .\, \neg\neg\neg a \supset \neg a\, .
\end{aligned}
$$

The solutions to 4.3 and 4.2 are clear without further calculation. One obtains the solution to 4.32 from 4.3 and 4.2 if one substitutes $\neg\neg a$ for b in 4.2.

If one adds the formula

$$\vdash .\, a \vee \neg a\, . \tag{1}$$

(in sentential logic, the principle of the excluded middle) to the *a priori* accepted collection B of formulas, one obtains a complete axiom system for classical sentential logic. In our problem interpretation the formula (1) reads as follows: to provide a general method for each problem a, either to give a solution to a, or to derive a contradiction from the assumption of such a solution! In particular, if the problem a consists of proving a statement, then one would need to possess a general method for each statement, either to prove it, or from it to derive a contradiction. Should our readers not consider themselves omniscient, they will readily determine that formula (1) cannot occur on the list of problems they have solved.

It is however noteworthy that the problem

$$4.8.\ \vdash .\, \neg\neg\, .\, a \vee \neg a\, .^{13}$$

[12] Heyting, I.

can be solved, as Heyting's calculations show.

The formula

$$(2) \qquad \vdash . \neg\neg a \supset a .$$

(in sentential logic, the principle of double negation) similarly cannot appear in our problem calculus, for otherwise from it one can derive the formula (1) with the help of (4.8).

So we see that, in contrast to Heyting's formulas of intuitionistic logic, already very simple formulas of classical sentential logic can no longer appear in our calculus of problems.

It should be noted that, if a formula $\vdash p$ in classical sentential logic is false, then the corresponding problem $\vdash p$ cannot be solved. One can, in fact, from such a formula $\vdash p$ with the help of previously accepted formulas and rules of the calculus derive the obviously contradictory formula $\vdash \neg p$[14].

§2.

The founding principle of the intuitionistic critique of logical and mathematical theories is the following: *Every non-content-free statement should refer back to one, or several, quite definite truth conditions which are accessible to our experience*[15].

If a is a general proposition of the form "every element of the collection K has property A" and, furthermore, K is infinite, then the negation of a, "a is false" does not satisfy the above principle. To avoid such circumstances, Brouwer gives a new definition of negation: "a is false" should mean "a leads to a contradiction". Thus the negation of a is transformed in to an *existential statement*: "There exists a chain of logical deductions which leads from the assumption of a to a contradiction". But Brouwer also subjected existential statements to a stringent critique.

For, from an intuitionistic standpoint, it makes simply no sense simply to say: "There is among the elements of a infinite collection K at least one with the property A", without exhibiting that element.

But Brouwer does not wish to throw existential statements completely out of mathematics. He merely explains, that one should not utter an existential statement, without giving a corresponding construction. On the other hand, for Brouwer an existential statement is no mere declaration that we have found a corresponding element in K. In the latter case, the existential statement would be false *before* the discovery of the element, and only *afterwards* true. So there emerges this quite particular sort of statement, which is to possess a content

[13] In sentential logic (4.8) represents Brouwer's proposition on the non-contradictory nature of the excluded middle.

[14] Cf. V. Glivenko, Bulletin of the Royal Academy of Belgium, 5^{th} series, vol. 15, p. 183, 1929.

[15] Cf. H. Weyl. On the new foundational crisis in mathematics, Mathematische Zeitschrift, vol. 10, p. 39, 1921. The whole subsequent investigation of negative and existential statements is connected to this work of Weyl.

which does not change through time, and yet which one can only enunciate under special conditions.

Naturally one can ask whether this special sort of statement is a fiction. There is in fact a problem here, namely "To find in the collection K an element with property A"; this problem has, in reality, a definite meaning, independent of the state of our knowledge; if one has *solved* this problem, *i.e.* if one has found such a corresponding element x, then one obtains an empirical statement: "Our problem is now solved. " Thus, what Brouwer understands as an existential statement, divides completely into two: the objective part (the problem) and the subjective (its solution). With that, no further object is needed to denote the actual meaning of an existential statement.

From this we should simply formulate the main result of the intuitionistic critique of negative statements: *For a general statement it is in general meaningless to regard its negation as a definite statement.* But then the characteristic feature of intuitionistic logic disappears, for now the principle of the excluded middle holds for all statements, whose negation has a meaning in general[16].

There then follows that one should regard the solution of problems, in and of itself, as the aim of mathematics (besides the proof of theoretical statements). And as shown in the first Section, the formulas of intuitionistic logic also receive a new meaning in the domain of problems and solutions[17].

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[16]But then there arises a new question: Which logical laws hold for statements whose negation has no meaning?

[17]Noted in proof. This interpretation of intuitionistic logic is intimately connected with the ideas which Heyting has developed in the latest volume of "Erkenntnis" (vol. 2, p. 106, 1931); in Heyting's version, though, no clear distinction is made between statements and problems.