On the calculating power of Laplace's demon

John Longley

LFCS, School of Informatics, University of Edinburgh, UK.
Email: jrl@inf.ed.ac.uk

Abstract. We discuss some of the choices that arise when one tries to make the idea of physical determinism more precise. Broadly speaking, 'ontological' notions of determinism are parameterized by one's choice of mathematical ideology, whilst 'epistemological' notions of determinism are parameterized by the choice of an appropriate notion of computability. We present some simple examples to show that these choices can indeed make a difference to whether a given physical theory is 'deterministic' or not.

Keywords: Laplace's demon, physical determinism, philosophy of mathematics, notions of computability.

1 Introduction

> *Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situations of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.* [19, chapter II]

In these now famous words, Laplace articulated his vision of an orderly, mechanistic universe whose history unfolds like clockwork according to fixed deterministic laws. In essence, this vision may be traced back at least to Democritus, and it has remained enormously influential down to the present time (see e.g. [25] for a modern incarnation). Philosophers still argue over whether or not the issue of physical determinism has any bearing on the problem of free will (see e.g. [6]). It is therefore very natural to ask how well Laplace's claim holds up in the light of our present-day understanding of science and mathematics.

Broadly speaking, the answer to this question will depend on two kinds of considerations. Firstly, it clearly depends on what the 'laws of physics' actually are: for example, some proposed formulations of quantum theory appear to allow for some kind of indeterminacy at the interface between 'quantum' and 'classical' levels, whilst others do not. Issues of this kind are clearly a matter for the physicists. Secondly, and less obviously, one can ask how exactly the Laplacian concept of determinism is to be made precise. It is this latter question that I wish to consider in this paper.

I will argue, drawing on ideas from computability theory and mathematical logic, that there are a whole range of different ways in which the idea of determinism might be understood. Some of the choices involved are purely technical in nature, whilst others touch on deeper philosophical issues. I will show, moreover that these choices can sometimes radically affect whether a physical theory is 'deterministic' or not, even in the case of very simple theories.

Laplace's imagery of a hypothetical predictive 'intelligence' (nowadays known as *Laplace's demon*) provides a valuable prop for the imagination. Roughly speaking, we will be asking exactly how the instantaneous state of the universe is supposed to be 'presented' to the demon (that is, what kinds of raw facts about this state he has access to), and exactly what kinds of 'analysis' — particularly what kinds of *infinitary* operations — he is supposed to be able to perform on this data. I hope to show how these considerations can make interesting and perhaps surprising differences to the conclusions that can be drawn.

1.1 Ontological versus epistemological determinism

As a first stab, we may broadly distinguish between two ways of interpreting Laplace's claim, which we call the *ontological* and the *epistemological* interpretation. The ontological version would say that given the present state of the universe (or of some closed physical system), there is, in fact, only one possible course of history starting from this state in which the laws of physics are upheld (whether or not we have any way of knowing what that history is). By contrast, an epistemological version would say that given knowledge of the present state, there is some way of *knowing* or 'working out' how the future will unfold.

The ontological and epistemological notions of determinism may be understood with reference to the mathematical notions of *truth* and *computability* respectively. Schematically, if histories are represented by mathematical functions from *Times* to *States*, ontological determinism claims that some sentence of the form

$$\forall s : \mathit{States},\ t : \mathit{Times}.\ \exists! h : \mathit{Times} \to \mathit{States}.\ h(t) = s \wedge \mathit{Laws\_Of\_Physics}(h) \quad (\dagger)$$

is *true*, where $\mathit{Laws\_Of\_Physics}(h)$ might say (for instance) that certain differential equations are satisfied at every point in space and time, and s specifies a boundary condition.[1] By contrast, epistemological determinism would claim that there is some kind of computable operation

$$\Phi : \mathit{States} \to (\mathit{Times} \to \mathit{States})$$

[1] Some care is needed over the status of the sentence (†). If it is understood purely as a mathematical statement about some *model* of physics, it does not succeed in saying anything about how the actual universe behaves. On the other hand, if it is understood as referring directly to physically real entities, it does not say what we want: since there is only one *actual* course of history, the uniqueness assertion becomes vacuous — we would really like h to range over all *mathematically* possible history functions. Our proposed solution is to understand (†) as a mathematical assertion, and to supplement it with the following statement, in which T, T' range

such that for any state s, the history $\Phi(s)$ correctly represents the evolution of the physical system starting from s. In connection with the ontological claim, one might imagine a demon so powerful that he can magically survey *all possible history functions* and pick out the one with the required property; for the epistemological version, one might imagine a more modest demon who makes predictions by following some kind of algorithmic procedure, perhaps involving idealized 'measurements' on the state s.

The *a priori* possibility that the behaviour of physical systems might be mathematically deterministic but not algorithmically computable in nature has been highlighted by Penrose [22, 23], who has furthermore suggested that physical laws of this kind might play an essential role in the science of consciousness.

1.2 Drawing finer distinctions

These two varieties of determinism presuppose, respectively, a notion of *mathematical truth* and a notion of *computability*. Discussions of determinism often implicitly assume that these are both unambiguous and unproblematic notions: surely in mathematics the notion of truth is absolute, and Church and Turing have provided us with the definitive notion of computability. However, an acquaintance with mathematical logic and computability theory would tend to suggest that things are not quite so simple.

On the one hand, the concept of mathematical truth certainly touches on deep philosophical issues. And since the idea of ontological determinism is itself of such philosophical interest, it is surely natural to ask what philosophical presuppositions this idea rests on. What metaphysical status does an assertion such as (†) about 'possible histories' really have? Different philosophies of mathematics would answer this question in very different ways. For example:[2]

- *Platonism* (the 'classical' view of mathematics) maintains that mathematical sentences like (†) do indeed have a definite truth-value independently of whether we can know what it is. This view involves a metaphysical commitment to a notion of truth not grounded in empirical or sensory experience.
- *Semi-constructivism* would subscribe only to a much more limited version of this idea: if $\phi(n)$ has a definite truth-value for each $n \in \mathbb{N}$, it is accepted that $\exists n.\phi(n)$ has a definite truth value. (This idea is embodied in the so-called 'Limited Principle of Omniscience'.) There is still a metaphysical commitment here, though it is more moderate than in the case of Platonism.

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over actual points in time and $S(T)$ is the actual state of the universe at time T.

$$\begin{array}{l}\forall h : \mathit{Times} \to \mathit{States}.\\ \quad (\forall t, T.\ \mathit{Models}(t,T) \Rightarrow \mathit{Models}(h(t), S(T))\ \Rightarrow\ \mathit{Laws\_Of\_Physics}(h)\end{array}$$

This at least isolates the *mathematical* content of determinism in our assertion (†).

[2] For a discussion of the main philosophical issues at stake (from an intuitionist perspective), see [7, chapter 7].

– *Constructivism* regards mathematical statements purely as expressions of what we can actually do or calculate; there is no reference to any independent notion of truth. Nothing entitles us to say $\exists x.\phi(x)$ other than *knowing* some suitable value of x. Under a constructivist reading of (†), our ontological notion of determinism might closely resemble an epistemological one.

Many further subdivisions and intermediate positions might be mentioned. We thus obtain a whole spectrum of interpretations of 'ontological determinism', involving varying levels of metaphysical commitment.

Regarding the question of *computability*, the familiar Turing notion is indeed generally accepted as the definitive notion for computations involving *natural numbers* or other finite entities that can be effectively coded by them. But what do we mean by computation where *infinite* entities are concerned, such as real numbers or continuous functions on the reals? Typically, many different answers to this question can be given, leading to several plausible but distinct notions of computability for such entities. Many of the issues, and possible choices, are discussed in [20], which focuses on computability at higher types over $\mathbb{N}$, an arena of particular interest within computer science. In other settings (e.g. higher types over $\mathbb{R}$, or the classical spaces of functional analysis), several computability notions have been proposed and studied (see e.g. [32, 36]), but we are still some way from seeing the overall picture. The crucial point here is that, in our characterization of epistemological determinism, the demand for a 'computable' operation Φ might be interpreted in many different ways.

It will be clear by now that there is considerable overlap between the ontological and epistemological notions. On the one hand, ontological determinism from a constructive standpoint can often be closely related to epistemological determinism based on some 'finitary' computability notion. Indeed, there is an extensive body of metamathematical work on using computability notions to model constructive formal systems (see e.g. [35]). On the other hand, ontological determinism from a non-constructive standpoint can often be related to epistemological determinism based on 'infinitary' computability notions. For instance, what 'exists' from a semi-constructivist standpoint is closely related to what is 'computable' in the presence of the existential quantifier $\exists : (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$ where $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$ (see [15]) — that is, what would be visible to a demon who could 'see all the natural numbers at once'.[3] In the presence of even more powerful infinitary operations, computability would approach classical notions of truth (see e.g. [29]).

All these considerations might seem rather arcane, and the suggestion that any of them might be of relevance to real physical theories might at first seem rather far-fetched. My purpose in the remainder of this paper is to present a selection of examples to show that these considerations really do make a difference to the question of determinism, even for simple physical theories.

[3] It is interesting to note that critiques of the Limited Principle of Omniscience sometimes take the form that it comes precariously close to presupposing the existence of such a 'demon'. See for example Wittgenstein [38, §352].

1.3 Infinities in the physical universe?

The questions of mathematical ideology discussed above, as well as the computability considerations we have mentioned, are closely bound up with the mathematical idea of 'the infinite'. Consequently, many of the issues we are discussing would trivialize if, in fact, the universe could be completely modelled by some discrete, finite mathematical structure. A brief discussion of this possibility is therefore in order; see also [24, chapters 3,33], [30, II.D], and [5].

The vast majority of successful physical theories in use today rely heavily on the calculus, which presupposes the mathematical idea of an infinitely subdivisible *continuum*. However, we do not know whether genuine continua — or infinities of any kind — actually occur anywhere in the physical world, and physicists have sometimes expressed unease at the seeming ontological extravagance of this assumption (see e.g. [11, pp. 57–8]). There have been several interesting attempts to put physics onto a more 'discrete' footing, but it would seem that this is not so easy to do, and most leading-edge physical theories still make extensive use of the mathematical continuum.

In view of this, it seems to us that it is interesting to explore the implications of the supposition that physical continua do exist. Even if, in the end, such an investigation served only to convince us of the implausibility of this supposition, this would still be a valuable outcome. Of course, investigations of this kind are perhaps of academic interest for physical theories that are already known not to hold 'all the way down', but they acquire an added dimension of significance for theories which are proposed as candidates for an 'ultimate' description of physical reality.

In this short article, my intention is not to map out a coherent programme of research, nor to consider current leading-edge physical theories in detail, but merely to collect together a few observations, based on known mathematical results, in order to illustrate the *kinds* of issues that can arise, and thus perhaps to indicate that the general area merits further exploration.

2 Determinism and the constructive continuum

First, I would like to explore some implications of adopting a strictly constructivist mathematical stance *à la* Bishop [3], by focusing on a childishly simple problem in Newtonian physics. A particle in one dimension is initially at rest, and no forces act on it: what happens to it?

Suppose we model this problem using a physical theory such as the following:[4]

$$\mathit{States} = \mathbb{R}, \quad \mathit{Times} = \mathbb{R}, \quad \mathit{Laws\_Of\_Physics}(h) \equiv \forall t.\, \dot{h}(t) = 0,$$
$$s = c \text{ (a constant)}, \quad t_0 = 0$$

[4] The standard Newtonian formulation of this problem would of course involve the second time derivative, but even the simplified version we give here will serve to illustrate our point.

The existence of suitable solutions, and even their computability (given c) is of course unproblematic; the issue is with the uniqueness part of (†). The problem is that, assuming $h(0) = c$ and $\mathit{Laws\_Of\_Physics}(h)$, we cannot *constructively* conclude that $h(t) = c$ for all t. This is because the Fundamental Theorem of the Calculus, saying that any continuous function has a unique antiderivative modulo an added constant, is not constructively valid.

One can understand the problem better by considering an alternative function h satisfying the above conditions within the universe of effective mathematics (which provides one possible model for constructive mathematics — see e.g. [14]). Such a solution is easy to construct using the well-known *Kleene tree* [15, §LII]. Looked at from a *classical* perspective, h is at most points a locally constant function, but one whose value jumps around, with discontinuities at a set of points homeomorphic to Cantor space. However, the values of t at which h is discontinuous are all non-computable reals — so seen from within the effective universe, h is continuous and has derivative 0 everywhere! Such a pathological history function is clearly ludicrous in physical terms, but the point is to ask how *precisely* we intend our theory to rule out such a possibility.

One might suppose that the problem could be fixed simply by strengthening our *Laws_Of_Physics* predicate in some way. But the same problem will beset *any* proposed predicate $\mathit{Laws\_Of\_Physics}(h)$ which is *local* in character:[5] that is, any predicate of the form $\forall t.\, L(h, t)$, where L satisfies

$$\forall h, h', t.\ (h, h' \text{ agree on some neighbourhood of } t) \ \Rightarrow\ (L(h,t) \Leftrightarrow L(h',t))$$

This is because the pathological solution above is locally just fine: at every point, it agrees locally with some globally constant function which we *do* want to allow.

There are, of course, many possible responses to this problem, e.g.:

1. Accept the non-determinism. (This would be silly: any theory that fails to predict that everyday objects do not jump about in this erratic fashion must be judged sorely deficient.)
2. Abandon the continuous model of time.
3. Abandon the locality principle. For example, we might postulate an additional physical law saying that h had to be *uniformly* continuous on any compact interval, a non-local property.[6]
4. Abandon strict constructivism, and admit some additional mathematical principle that allows us to deduce the uniqueness of h. One minimal such principle would be a weak (double-negation sanitized) version of König's Lemma for binary trees (cf. [31, Chapter IV]), which can certainly be justified on the philosophical premises of semi-constructivism.[7]

[5] The idea that the laws of physics ought to be local in character seems quite deeply ingrained in the informal conception of a 'mechanistic' universe. For an illuminating discussion of the 'locality principle' from a physicist's perspective, see [11, Chapter 2].

[6] This is in fact the notion of continuity adopted by Bishop-style constructivists in order to obtain a viable theory.

[7] Or, for that matter, on the premises of Brouwerian intuitionism.

One impression that emerges from this situation is that there is some kind of trade-off between the strength of the *physical* assumptions (as in 3) and that of the *mathematical* (or metamathematical) assumptions (as in 4) needed to conclude determinism. It does not pay to be too parsimonious on both fronts.

3 Finite dimensional systems

From a semi-constructivist or Platonist standpoint, it seems that for physical systems with *States* $= \mathbb{R}^n$ governed by ordinary differential equations (e.g. n-body problems), both ontological and epistemological determinism are relatively unproblematic. Indeed, there is a now well-established canonical computability notion for total functions $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ (first introduced by Lacombe [17] and Grzegorczyk [13]), and this notion appears to suffice for predicting behaviour in all cases of physical interest.

The essence of this notion of computability is that one can compute the output $f(\mathbf{x})$ to within any desired $\epsilon > 0$ if one knows the input $\mathbf{x}$ to within some $\delta > 0$ dependent on ϵ and $\mathbf{x}$. (In particular, all computable functions are continuous). We can therefore think in terms of a demon equipped with an infinite sequence of measuring devices of increasing resolving power, who is able to make affirmative observations on the state corresponding to *open subsets* of $\mathbb{R}^n$, and follows some effective procedure for processing the results of such observations. The set of all observationally affirmable properties that hold at time 1 (say) can then be computably determined from the set of all affirmable properties at time 0.

However, even in this relatively unproblematic setting, a couple of caveats need to be made. The first concerns physical systems whose behaviour exhibits singularities — whose state at time 1 can be discontinuous in the state at time 0. One example (described in [22, chapter 5]) is the collision problem for three elastic billiard balls: the classical physical theory does not determine what happens if the three balls collide at *exactly* the same point in time. At the very least, one should here modify one's claim of determinism to a conditional statement involving a computable *partial* function $\mathbb{R}^{n+1} \rightharpoonup \mathbb{R}^n$. It is fair to add here that the relationships between candidate definitions of computability for partial functions on the reals remain to be fully clarified.

A second caveat is a rather technical one concerning the notion of effective procedure involved: the demon had better not be following a program in a deterministic, *sequential* programming language. More specifically, imagine that once the demon has decided to test for some property like '$x > \frac{1}{2}$', he is committed to obtaining an answer before he can proceed with anything else. If indeed $x > \frac{1}{2}$, the demon will eventually discover this by means of a sufficiently precise measurement; but if $x = \frac{1}{2}$, he will be side-tracked into making measurements of increasing precision forever.[8] The problem can be overcome if the demon is allowed to use a 'parallel conditional' operator of the kind used in [9].

[8] This is reminiscent of the fact that even functions as simple as addition are not computable in the sequential version of Real PCF [10].

4 Infinite dimensional systems

For physical systems with infinite state spaces (e.g. continuously varying fields governed by partial differential equations), the picture gets considerably more interesting. As an example, we will consider the *wave equation* for a scalar field $\psi(\mathbf{x}, t)$ in three space dimensions: $\nabla^2\psi = \ddot{\psi}$. We take the set *States* to be some space of functions $s : \mathbb{R}^3 \to \mathbb{R}^2$: the first component of $s(\mathbf{x})$ gives the instantaneous value of ψ at $\mathbf{x}$, while the second component gives its time-derivative $\dot{\psi}$. We are therefore interested in the computability or otherwise of an operation of type $[\mathbb{R}^3 \to \mathbb{R}^2] \to [\mathbb{R}^4 \to \mathbb{R}^2]$.

The theory of second-order computability over the reals is far from trivial, and there are various choices available to us. First, let us briefly consider one choice that will *not* work well here: namely, Kleene-style higher-type computability in the spirit of [15]. In this approach, functions are treated as *oracles*, so that the only way to extract information about a function $[\mathbb{R}^3 \to \mathbb{R}^2]$ would be to apply it to particular values $\mathbf{x}$. Suppose then that we consider the style of computation envisaged in Section 3 augmented with such oracle calls. This would mean that a computation of the value of $\psi(\mathbf{0}, 1)$ (say) to within some $\epsilon > 0$ would only be able to interrogate the initial state s at finitely many points before returning an answer. It is easy to see that this is not enough, since the true value of $\psi(\mathbf{0}, 1)$ cannot be determined even up to ϵ from such a finite sample.

Our demon must therefore somehow be able to observe properties of s that pertain to whole *regions* of $\mathbb{R}^3$, rather than simply its value at particular points. One could of course endow the demon with the ability to survey regions by adding a suitable quantifier such as $\exists_{\mathbb{R}}$ to its repertoire, but this would seem like overkill. It seems more interesting to regard certain kinds of observations over regions as 'atomic', and ask whether some finitary style of computation involving these observations will suffice.

4.1 The Pour-El/Richards approach

A more suitable approach involves the theory of computability for Banach spaces developed by Pour-El and Richards [27, 28], who have moreover made a particular study of the wave equation in their setting. In this theory, one first axiomatizes the notion of a *computability structure* on a Banach space X, consisting of a set of *computable sequences* $\mathbb{N} \to X$ with certain closure properties. An element $x \in X$ is considered computable if $x, x, x, \ldots$ is a computable sequence.

Although computability structures are defined axiomatically, it turns out for quite general reasons that, for all naturally occurring spaces X, there is only one reasonable choice of computability structure. This gives the theory the attractive feature that the notion of computability is *intrinsic* for such spaces. However, this computability notion is sensitive to the choice of *norm* on the space: thus, for instance, there exist continuous functions $s : \mathbb{R}^3 \to \mathbb{R}^2$ with compact support which are non-computable if regarded as elements of $C(\mathbb{R}^3, \mathbb{R}^2)$ (the space of bounded continuous functions with the supremum norm), but computable if regarded as elements of $L_2(\mathbb{R}^3, \mathbb{R}^2)$. Insofar as the choice of norm is up to us rather than given 'by nature', we therefore have a range of possible computability notions even at the level of individual states.[9]

Next, consider linear maps $T : X \to Y$ between Banach spaces with computability structures. A major theorem of [28] asserts that, under modest conditions, T maps computable sequences to computable sequences iff T is *bounded*. In fact it is very reasonable to regard such maps T as the *computable maps* $X \to Y$, as is shown by results of [32, 33]. Specifically, the set of computable elements of a Banach space with computability structure can naturally be endowed with a *computable representation* in the sense of Weihrauch [36] (or alternatively with an effective domain representation in the sense of Scott and Ershov). It can then be shown that the linear maps that satisfy the Pour-El/Richards conditions are precisely those whose action on computable elements is computable in the Weihrauch sense (or alternatively in the sense of effective domain theory).

In the case of the supremum norm, these notions have especially good credentials. A natural and robust notion of second order computability over the reals has been studied in [2], where several non-trivially equivalent characterizations are given. Here one is able to perform computations on first order functions f by making use of 'compact-open observations' about them: '$f(x) \in U$ for all $x \in K$', where U is open and K is compact. This is one way to make precise the idea of 'atomic observations over regions' mentioned earlier. Now let X be the Banach space $C(\mathbb{I}^3, \mathbb{R}^2)$ (we here replace $\mathbb{R}$ by the unit interval $\mathbb{I}$ to avoid some annoying technical complications). Then the computable maps $X \to X$ turn out to be precisely the linear maps that are computable in the sense of [2].

Given the good credentials of this notion, it is perhaps all the more surprising that the solution operators for the wave equation are *not* computable in this sense. For simplicity, let $X = C(\mathbb{R}^3, \mathbb{R}^2)$, and let $T : X \to X$ be the operator that maps a wave state at time 0 to the wave state at time 1 that results from it. It was shown in [26] that $T : X \to X$ is not computable — in fact, that there is even a computable state s such that the first component of $T(s)(\mathbf{0})$ is a non-computable real. An important aspect of the example given in [26] is that though the first component of s is computable, its space-derivative is non-computable.

What about the computability notions arising from spaces with other norms? One important example considered in [28] is the *energy norm* $\| - \|_E$ given by:

$$\| \langle s_0, s_1 \rangle \|_E^2 \quad = \quad \iiint_{\mathbb{R}^3} |\nabla s_0|^2 + s_1^2$$

In physical terms, this corresponds to the total amount of energy present in a wave state, and it satisfies the axioms required for a norm. The fact that energy is conserved as the wave propagates says that the operator T is bounded by 1, and is therefore a computable map. The wave equation thus provides a good example of a physical theory for which the question of epistemological determinism is sensitive to the computability notion adopted.

[9] Interestingly, in the case of *quantum mechanical* systems this issue seems not to arise, since the physical theory itself makes essential use of an inner product which gives us a preferred choice of norm. See [1].

The precise conditions under which the solution operator is or is not computable have been investigated in some detail by Weihrauch and Zhong [37]. As these authors point out, different choices of norm (or topology) correspond to different notions of 'possible affirmative observation' on states, and it is a matter of physics to investigate whether any of these choices correspond to what is observable by means of 'idealized physical measurements' of some kind.

Let us suppose that the class of 'physically realizable' affirmative observations (in some idealized sense) corresponded precisely to the topology induced by some norm $\| - \|_{obs}$. The effective domain of closed balls for this norm (say) would then constitute an *epistemic* (i.e. information-theoretic) model of the physical system, and states (considered as functions $\mathbb{R}^3 \to \mathbb{R}^2$) would appear as maximal consistent sets of affirmable observations. However, a more minimalist ontology might then reject these 'ideal' elements in favour of a purely finitary theory of possible observations, in which idealized mathematical points would be abandoned in favour of a theory of intervals or regions, somewhat in the spirit of [8]. Indeed, one can conceive that this might offer one way of putting the whole physical theory on a more 'constructive' or 'finitary' footing as suggested in [30].

If it turned out that T were computable with respect to $\| - \|_{obs}$, we would be in the very pleasing situation that the set of potentially observable facts at time 1 was computably determined by the set of such facts at time 0 — and this would also systematically explain why the ontologically underlying non-computable values of the wave function ψ (on computable arguments, for computable initial states) can never surface at the level of what is measurable. However, if (as the author suspects is more likely), T were not computable with respect to $\| - \|_{obs}$ we would face a dilemma similar to the one encountered at the end of Section 2: either certain kinds of state information must be considered as ontologically real though they are not observable even by approximation, or the future state cannot be effectively predicted from the present state.

Conclusion and acknowledgements

In this short paper we have barely scratched the surface of our proposed area of investigation, and many of the issues we have highlighted demand much more detailed discussion than we have provided. Nevertheless, we hope that our selection of examples has persuaded the reader that the task of elucidating different notions of determinism is interesting and worthwhile, and that this may prove to be a fruitful area of interaction between physics and computability theory.

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APPENDIX: A thought experiment concerning non-computability in the physical world

We here present a simple thought experiment in connection with the question of whether non-computability ever manifests itself in the physical world. Though this stands somewhat apart from our present inquiry into possible interpretations of determinism, it is clearly relevant to whether the laws of physics *are* deterministic in a strong epistemological sense, and provides another example of the application of ideas from computability theory to such questions.

It is tempting to imagine that — because non-computability is an infinitary property that cannot be detected from any finite sample of data, and because every continuous function on $[0, 1]$ can be approximated arbitrarily closely by a computable one, and so on — it cannot possibly make any kind of observable difference whether the laws of physics are computable or not. The following experiment suggests one possible sense in which this is not the case.

The experiment involves the *Lacombe tree* [18], a variation on the Kleene tree which deserves to be better known. The Lacombe tree is a computable binary tree T (that is, a decidable prefix-closed set of finite sequences over $\{0, 1\}$) such that

- every *computable* infinite sequence over $\{0, 1\}$ eventually exits from T; but
- classically, the set of infinite sequences that eventually exit from T has some measure $m < \frac{1}{2}$ within $\{0, 1\}^{\mathbb{N}}$.

One may therefore imagine setting up 100 instances of the following apparatus. Some physical device not known to have computable behaviour (perhaps a Geiger counter) is set up to generate a stream of binary digits, which are fed into a computer. At each stage, the computer tests whether the finite sequence received so far is a node of T. If, at some stage, the sequence is found to have exited from T, a light is turned on.

After allowing the experiment to run for some finite period of time, we return and count how many of the lights have come on. If the answer is fewer than $m.100$, we cannot conclude anything — perhaps we have just not waited long enough for the other sequences to exit yet. However, if the answer is 95 (say), this provides good evidence that the infinite sequences generated by our devices are *not* being drawn at random from the classical set $\{0, 1\}^{\mathbb{N}}$, but from some more restricted set, perhaps the set of computable sequences. If one supposes that the choice is between 'all sequences' and 'the computable sequences', we thus have a probabilistic semi-decision test for whether the behaviour of our physical devices is computable. (Interesting further questions arise if we also consider the possibility of other sets, such as the set of *hyperarithmetic* sequences.)

We hasten to add that this experiment would be of no use in practice, in view of the hyper-astronomical length of time one would have to wait for a significant number of lights to come on.

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