The Complexity of Nash Equilibria and Fixed Points of Algebraic Functions

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Appetizer

What is the complexity of each of the following search problems:

a. (Nash, 1950) Given a finite game, and $\epsilon > 0$, compute a vector x' (a mixed strategy profile) within distance ϵ of some (exact) Nash Equilibrium.

b. (Shapley, 1953) Given an instance of Shapley's stochastic game, and $\epsilon > 0$, approximate the *value* of the game to within distance ϵ.

Note:

`Parity-Games` $\leq_p$ `Mean-Payoff-Games` $\leq_p$ `Simple-Stochastic-Games` $\leq_p$ $\ldots$ $\leq_p$ `Approximate-Shapley's-Stochastic-Games`

c. (Kolmogorov, 1947) Given a multi-type Branching Process, and $\epsilon > 0$, approximate its *extinction probabilities* within distance ϵ.

Question: What do these three problems have to do with each other?

Hint: They are all fixed point problems for algebraically defined functions.
Respectively:

a. *Brouwer*

b. *Banach*

c. *Tarski*

But are they related in terms of computational complexity? Yes.

Outline of talk

- Background: Games, Nash Equilibria, Brouwer Fixed Points.
- Weak vs. Strong approximation of Fixed Points.
- Scarf's classic algorithm, and its complexity implications.
- The complexity class PPAD, and weak approximation.
- Hardness of strong approximation: square-root-sum & arithmetic circuits.
- A new complexity class: **FIXP**. Nash is FIXP-complete.
- linear-FIXP = PPAD.
- Other FIXP problems: price equilibria, stochastic games, branching processes...
- Conclusions and future challenges.

Finite Games

A finite (normal form) *game*, Γ, consists of:

1. A set $N = \{1, \ldots, n\}$ of players.
2. Each player $i \in N$ has a finite set $S_i = \{1, \ldots, m_i\}$ of (pure) *strategies*.

   Let $S = \prod_{i=1}^{n} S_i$.
3. Each player $i \in N$, has a *payoff (utility) function* $u_i : S \mapsto \mathbb{Q}$.

mixed strategies, expected payoffs, etc.

- A *mixed strategy*, $x_i = (x_{i,1}, \ldots, x_{i,m_i})$, for player i is a probability distribution over S_i.

  A *profile* of mixed strategies: $x = (x_1, \ldots, x_n)$

  Let X denote the set of all profiles.

- The *expected payoff* for player i:

$$U_i(x) = \sum_{s=(s_1,\ldots,s_n)\in S} \left(\prod_{k=1}^{n} x_{k,s_k}\right) u_i(s)$$

- Let x_{-i} denote everybody's strategy in x except player i's.

  Let $(x_{-i}; y_i)$ denote the new profile: $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

Nash Equilibria

A mixed strategy profile x is called:

- a *Nash Equilibrium* if:

  $\forall\ i$, and all mixed strategies y_i: $U_i(x) \geq U_i(x_{-i}; y_i)$

  I.e.: *No player can increase its own payoff by unilaterally switching its strategy.*

- a ϵ-*Nash Equilibrium*, for $\epsilon > 0$, if:

  $\forall\ i$, and all mixed strategies y_i: $U_i(x) \geq U_i(x_{-i}; y_i) - \epsilon$

  I.e.: *No player can increase its own payoff by more than* ϵ *by unilaterally switching its strategy.*

Theorem (Nash 1950) *Every finite game has a Nash Equilibrium.*

Nash's proof

Brouwer's fixed point theorem: A continuous function $F : D \mapsto D$ from a compact convex set $D \subseteq \mathbb{R}^m$ to itself has a fixed point: $x^* \in D$, s.t. $F(x^*) = x^*$.

Nash showed the NEs of a finite game, Γ, are precisely the fixed points of the following Brouwer function $F_\Gamma : X \mapsto X$:

$$F_\Gamma(x)_{(i,j)} = \frac{x_{i,j} + \max\{0, g_{i,j}(x)\}}{1 + \sum_{k=1}^{m_i} \max\{0, g_{i,k}(x)\}}$$

where $g_{i,j}(x) \doteq U_i(x_{-i}; j) - U_i(x)$.

Note: $g_{i,j}(x)$ are polynomials in the variables in x, and they measure:

"how much better off would player i be if it switched to pure strategy j?"

A basic computational question

What is the complexity of the following search problem:

("Strong") ϵ-approximation of a Nash Equilibrium:

Given a finite (normal form) game, Γ, with 3 or more players, and given $\epsilon > 0$, compute a rational vector x' such that there is some (exact!) Nash Equilibrium x^* of Γ so that:

$$\|x^* - x'\|_\infty < \epsilon$$

Note: This is NOT the same thing as asking for an ϵ-Nash Equilibrium.

Weak vs. Strong approximation of Fixed Points

- 2-player finite games always have rational NEs, and there are algorithms for computing an exact rational NE in a 2-player game (Lemke-Howson'64).
- For games with ≥ 3 players, all NEs can be irrational (Nash,1951).

  So we can't hope to compute one "*exactly*".

Two different notions of ϵ-approximation of fixed points:

- (**Weak**) Given $F : \Delta_n \mapsto \Delta_n$, compute x' such that: $\|F(x') - x'\| < \epsilon$.
- (**Strong**) Given $F : \Delta_n \mapsto \Delta_n$, compute x' such that there exists x^* where $F(x^*) = x^*$ and $\|x^* - x'\| < \epsilon$.

Scarf's classic algorithm

Scarf (1967) gave a beautiful algorithm (refined by Kuhn and others) for computing (weak!) ϵ-fixed points of a given Brouwer function $F : \Delta_n \mapsto \Delta_n$:

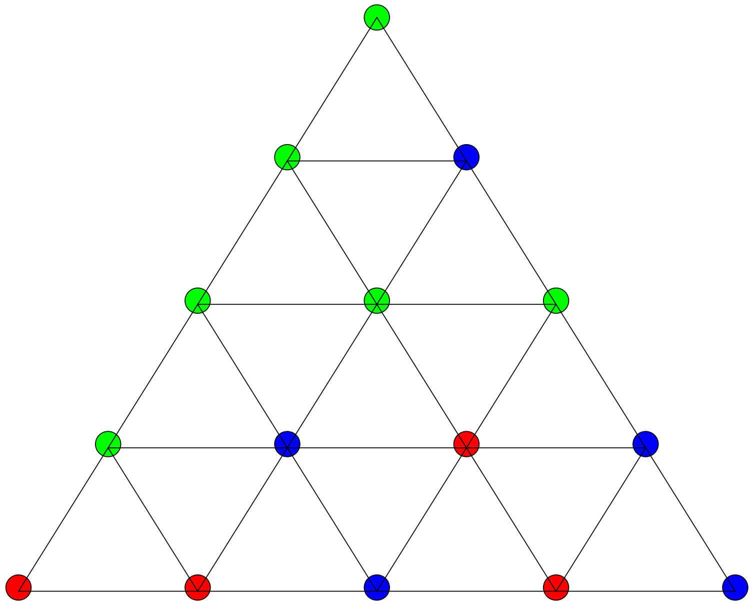
1. Subdivide the simplex Δ_n into "small" subsimplices of diameter $\delta > 0$ (depending on the "modulus of continuity" of F, and on $\epsilon > 0$).

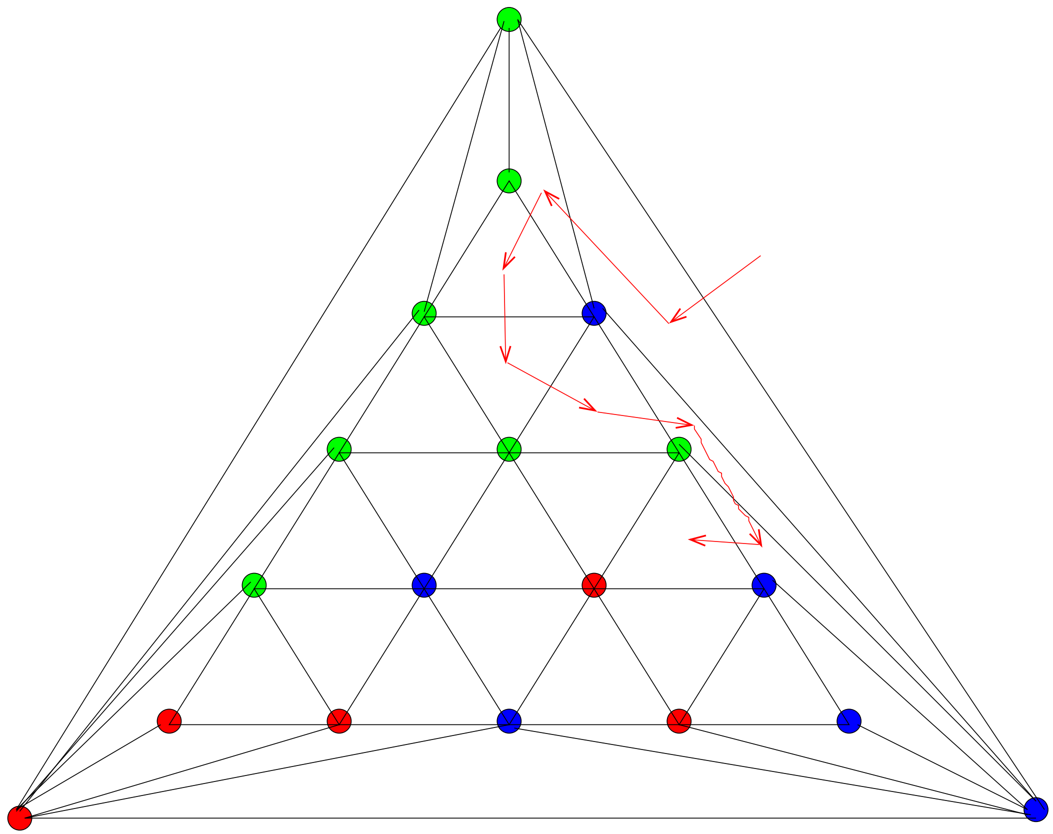
2. Color every *vertex*, $\mathbf{z}$, of every subsimplex with a color i such that $\mathbf{z}_i > 0$ & $F(\mathbf{z})_i \leq \mathbf{z}_i$.

3. By **Sperner's Lemma** there must exist a panchromatic subsimplex. (And the proof provides a way to "navigate" toward such a simplex.)

4. *Fact:* If $\delta > 0$ is chosen such that $\delta \leq \epsilon/2n$ and $\forall x, y \in \Delta_n$, $||x - y||_\infty < \delta \Rightarrow ||F(x) - F(y)||_\infty < \epsilon/2n$, then all the points in a panchromatic subsimplex are weak ϵ-fixed points.

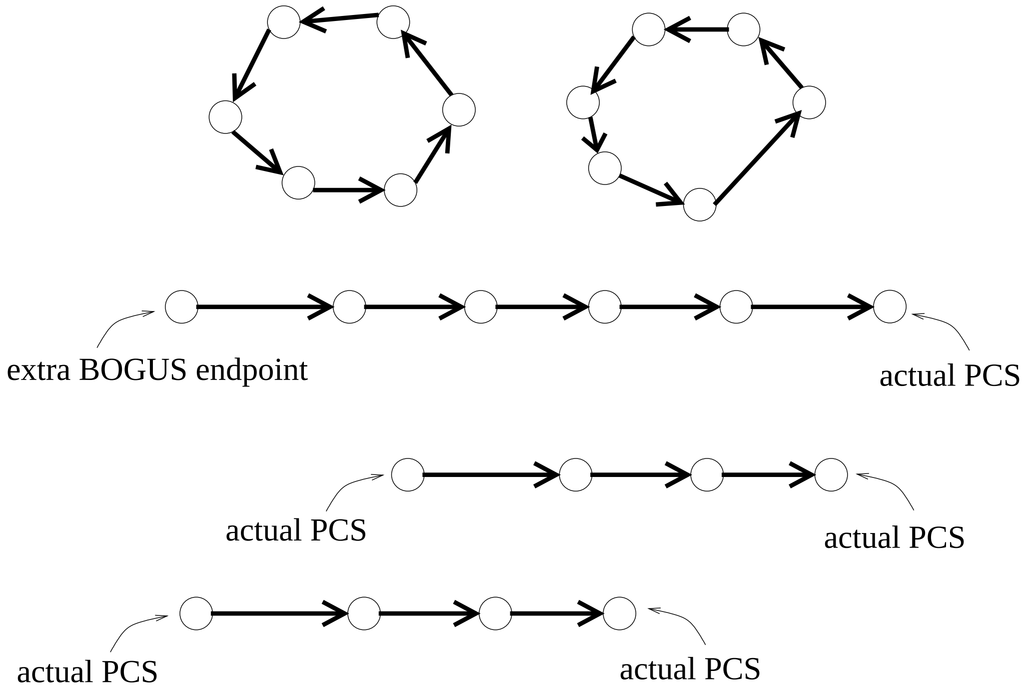
   (They need NOT in general be anywhere near an actual fixed point.)





The underlying "directed lines" parity argument in Scarf's algorithm

(Same combinatorial argument used by [Lemke-Howson'64] for 2-player Nash.)



Implicit assumptions: when is Scarf's algorithm applicable?

To use Scarf's algorithm for computing a weak ϵ-fixed point (in a reasonably efficient way) we are making several implicit assumptions. Suppose $F : \Delta_n \mapsto \Delta_n$ is given to us in a (unspecified) form that requires m bits to describe.

1. $F(x)$ should be *polynomial-time computable* for given rational vector x. I.e., the time to compute $F(x)$ should be polynomial in both m and the encoding size of x. (Otherwise, how can we compute colors of vertices efficiently?)
2. We should have a "tractable" simplicial subvidivision of Δ_n.
   In particular, the subsimplices and their vertices, z, must have encoding size polynomial in m and $size(\epsilon)$. (Otherwise, again, how can we compute $F(z)$ efficiently?) And the simplicial subdivision must yield efficient algorithms (P-time in m and $size(\epsilon)$) for both starting at the extra bogus endpoint subsimplex, and for traversing "on the fly" a single directed edge of the (implicit) line graph whose nodes are subsimplices.

another key assumption.....

3. Finally, $F(x)$ should be *polynomially continuous*, meaning, that there is a polynomial $q(r)$ such that for a given $\epsilon > 0$, we can choose $\delta = 1/2^{q(m+size(\epsilon))}$, such that $\forall x, y \in \Delta_n$, $\|x - y\| < \delta \Rightarrow \|F(x) - F(y)\| < \epsilon$.

   (Note: since F is continuous on a compact set Δ_n, it is *uniformly* continuous.)

These assumptions (1. – 3.) do not guarantee that Scarf's algorithm will run in P-time. They just guarantee that each step (each edge traversal) of Scarf's algorithm can be carried out in P-time, and that it will eventually halt (after potentially exponentially many traversal steps in the encoding size m and in $size(\epsilon)$, because there can be exponentially many subsimplices), and will produce a panchromatic subsimplex such that every point inside that subsimplex is a weak ϵ-fixed point of F.

ϵ-NEs are weak ϵ-fixed points

Fact: *For finite games, Γ, computing an ϵ-NE is P-time equivalent to computing a Weak ϵ-fixed point of Nash's function F_Γ.*

Thus, to compute an ϵ-NE, we can simply apply Scarf's algorithm to F_Γ.

The functions F_Γ satisfy all the implicit assumptions for applicability of Scarf's algorithm: they are polynomially continuous, polynomial-time computable, and furthermore appropriate "tractable" simplicial subdivisions are well known for the compact convex domain X of mixed strategy profiles (i.e., for cartesian products of n-simplices).

Question: *What does this tell us about the complexity of computing an ϵ-NE?*

The complexity class PPAD

Papadimitriou (1992) defined **PPAD**, based on the "directed line" parity argument, to capture (approximate) Nash and (approximate) Brouwer, etc...

Definition: **PPAD** is the class of search problems polynomial-time reducible to:

Directed line endpoint problem: Given two boolean circuits, S ("Successor") and P ("Predecessor"), each with n input bits and n output bits, such that $P(0^n) = 0^n$, and $S(0^n) \neq 0^n$, find a n-bit vector, $\mathbf{z}$, such that either: $P(S(\mathbf{z})) \neq \mathbf{z}$ or $S(P(\mathbf{z})) \neq \mathbf{z} \neq 0^n$.
(By the directed line parity argument such a $\mathbf{z}$ exists (for inconsistent P and S it exists trivially).)

PPAD lies somewhere between (the search problem versions of) P and NP.

By Scarf's algorithm, computing a ϵ-NE is in PPAD.

Can we do better?

Computing ϵ-NEs is already as hard as all of PPAD:

Theorem:

1. [Daskalakis-Goldberg-Papadimitriou'06][Chen-Deng'06]:

   Computing a ϵ-NE for a 3 player game is PPAD-complete.

2. [Chen-Deng'06]:

   Computing an exact (rational) NE for a 2 player game is PPAD-complete.

But what if we want to approximate exact NEs for games with ≥ 3 players and to approximate exact fixed points?

I.e., what if we want to do *strong* approximation of fixed points?

(**Warning:** Scarf's algorithm does not in general yield Strong ϵ-fixed points.)

Why care about strong approximation of fixed points?

- It can be argued (as Scarf (1973) implicitly did) that for many applications in economics weak ϵ-fixed points of Brouwer functions are sufficient.

- However, many important problems boil down to a fixed point computation for which weak ϵ-FPs are useless, unless they also happen to be strong ϵ-FPs.

  Examples:

  –Shapley's Stochastic Games;
  –Condon's (1992) Simple Stochastic Games;
  –Kolmogorov's multi-type Branching Processes;

  (and *Recursive Markov Chains*, and *Recursive Stochastic Games*, ...)

A basic upper bound for Strong ϵ-approximation of Nash

Fact: *Given game* Γ *and* $\epsilon > 0$*, we can Strong* ϵ*-approximate a NE in* **PSPACE**.

Proof: For Nash's functions, F_Γ, the expression

$$\exists \mathbf{x}(\mathbf{x} = F_\Gamma(\mathbf{x}) \wedge \mathbf{a} \leq \mathbf{x} \leq \mathbf{b})$$

can be expressed as a formula in the *Existential Theory of Reals* (ETR). So we can Strong ϵ-approximate an NE, $x^* \in \Delta_n$, in **PSPACE**, using $\log(1/\epsilon)n$ queries to a PSPACE decision procedure for ETR ([Canny'89],[Renegar'92]).
(These are deep, but thusfar impractical algorithms.) ∎

Can we do better than **PSPACE**?

The Square-Root Sum problem

The square-root sum problem (**Sqrt-Sum**) is the following decision problem:

Given $(d_1, \ldots, d_n) \in \mathbb{N}^n$ and $k \in \mathbb{N}$, decide whether $\sum_{i=1}^{n} \sqrt{d_i} \leq k$.

It is known to be solvable in PSPACE.

[Allender, Bürgisser, Kjeldgaard-Pedersen, Miltersen, 2006] improved this to:

the 4th level of the *Counting Hierarchy*: $P^{PP^{PP^{PP}}}$.

Open problem ([GareyGrahamJohnson'76]) whether it is solvable even in NP.

(In particular, whether exact Euclidean-TSP is in NP hinges on this.)

Sqrt-Sum $\leq_p$ approximation of actual NE

Theorem: *Any non-trivial approximation of an actual NE solves* **Sqrt-Sum**.

More precisely:

For every $\epsilon > 0$, **Sqrt-Sum** *is P-time reducible to the following problem:*

Given a 3-player (normal form) game, Γ, with the property that:

1. *in every NE, player 1 plays exactly the same mixed strategy, x_1^*, and*
2. *the probability, $x_{1,1}^*$, with which player 1 plays it first pure strategy is either:*

(a.) 0 , *or* *(b.)* $\geq (1 - \epsilon)$,

decide which of (a.) or (b.) is the case.

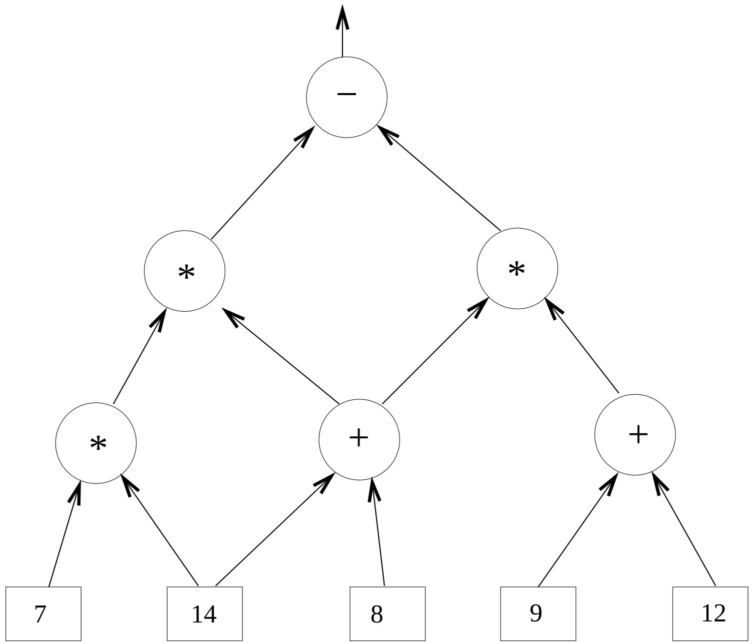
A harder arithmetic circuit decision problem

[Allender et al'06] reduced **Sqrt-Sum** to the following (which they showed lies in the *Counting Hierarchy*):

PosSLP: Given an *arithmetic circuit* (Straight Line Program) over basis $\{+, *, -\}$ with integer inputs, decide whether the output is > 0.

Every *discrete* decision problem solvable in P-time in the unit-cost arithmetic RAM model in P-time, i.e., in the discrete, rational Blum-Shub-Smale class $\mathbf{P}_{\mathbb{R}}$, is P-time (Turing) reducible to **PosSLP**.
So, **PosSLP** captures discrete problems in $\mathbf{P}_{\mathbb{R}}$.

(Note: testing $= 0$ for such arithmetic circuits (much easier than PosSLP) is already a well-known open problem. It is equivalent ([ABKM'06]) to *polynomial identity testing* (known to be in coRP).)



Theorem:

PosSLP *is P-time reducible to Strong approximation of 3-player NEs.*

More precisely, it reduces to the same 0 vs. $(1 - \epsilon)$ choice problem as before.

Question: How far can an ϵ-NE be from an actual NE?

Answer: Very far.

A seemingly contrary fact:

Fact: *For every continuous function $F : \Delta \mapsto \Delta$, and every $\epsilon > 0$, there exists a $\delta > 0$, such that a weak δ-fixed point of F is a strong ϵ-fixed point of F.*

But this is a non-constructive fact!
It uses a compactness argument. (Bolzano-Weierstrass.)

From a quantitative, computational perspective, it is certainly not the full story:

Theorem: *For every n, there exists a 4-player game Γ_n of size $O(n)$ with an ϵ-NE, x', where $\epsilon = \frac{1}{2^{2^{\Omega(n)}}}$, and yet x' has distance 1 (in l_∞) to any actual NE.*

Same holds for 3 players, but with distance 1 replaced by distance $(1 - 2^{-poly})$.

A new complexity class: FIXP

Consider the following class of fixed point problems:

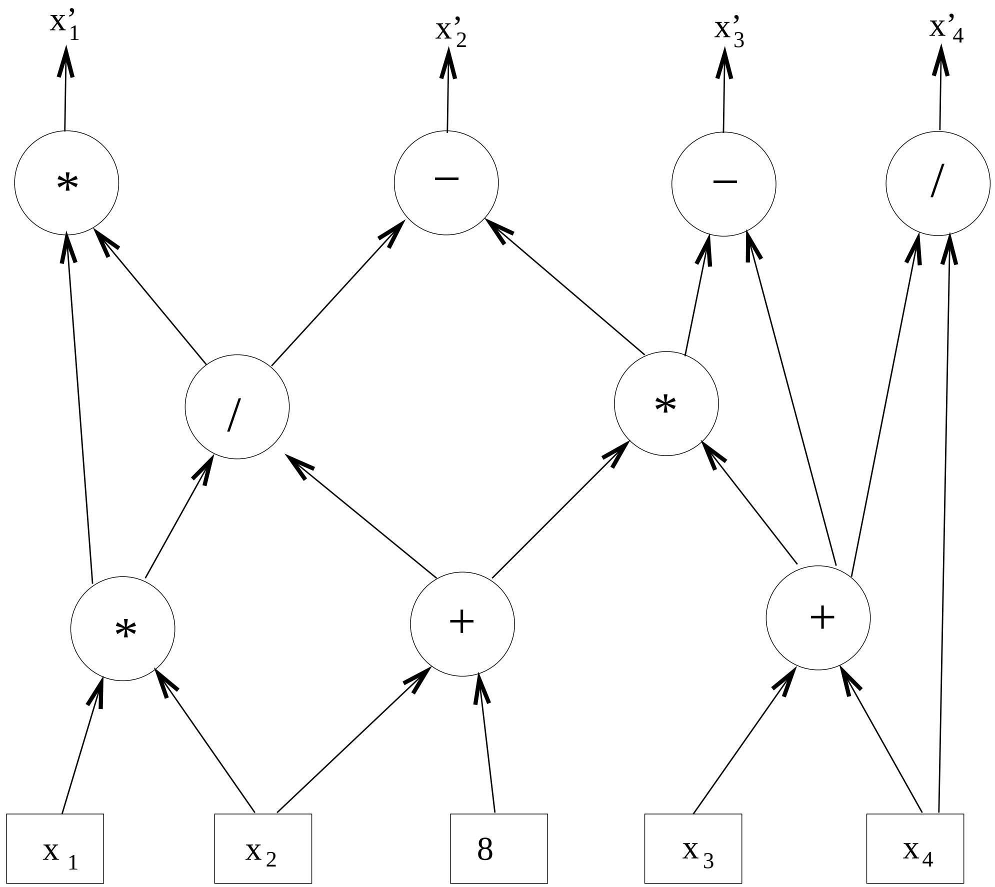
Input: algebraic circuit (straight-line program) over basis $\{+, *, -, /, \max, \min\}$ with rational constants, having n input variables and n outputs, such that the circuit represents a continuous function $F : [0,1]^n \mapsto [0,1]^n$.

(The domain $[0,1]^n$ can be allowed to be much more general. See our full paper.)

Output: Compute (or strong ϵ-approximate) a fixed point of F.

We close these problems under suitable P-time reductions.

Call the resulting class **FIXP**.


x'1
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x 1
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x 3
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Nash is FIXP-complete

Theorem *Computing a 3-player Nash Equilibrium is* **FIXP**-*complete.*

It is complete in several senses:

- In terms of "exact" (real valued) computation;

- In terms of strong ϵ-approximation,

- An appropriate "decision" version of the problem: Given a game,Γ, rational value $q \in \mathbb{Q}$, and coordinate i: if for all NEs x^*, $x^*_i \geq q$, then *"Yes"*; if for all NEs x^*, $x^*_i < q$, then *"No"*. Otherwise, any answer is fine.

Completeness holds under very restrictive polynomial-time (real valued) search problem reductions where the "solution recovery" function g is linear.

Very brief sketch of some proof ingredients

- Suppose we could create a (3-player) game such that, in any NE, Player 1 plays strategy A with probability $> 1/2$ iff $\sum_i \sqrt{d_i} > k$ and with probability $< 1/2$ iff $\sum_i \sqrt{d_i} < k$. (Suppose equality can't happen.)
- Add an extra player with 2 strategies, who gets high payoff if it "guesses correctly" whether player 1 plays pure strategy A, and low payoff otherwise.

  In any NE, the new player will play one of its two strategies with probability 1.

  Deciding which solves SqrtSum.
- What about equality? We don't have to worry about it because $\sum_i \sqrt{d_i} = k$ is P-time decidable ([Borodin-Fagin-Hopcroft-Tompa'85]).

A key ingredient in our proofs

Two beautiful gems by Bubelis:

1. (Bubelis, 1979) *Every real algebraic number can be "encoded" in a precise sense as the payoff to player 1 in a unique NE of a 3-player game.*

2. (Bubelis, 1979) *There is a general polynomial-time reduction from n-player games to 3-player games*.

   Such that you can easily recover a (real valued) NE of the n-player game as a linear function of a given NE in the resulting 3-player game.

Many details in the proof of FIXP-completeness:

- A series of transformations to get circuits into a "normal form" with additional "conditional assignment gates".
- Transform circuit to a game with a large (but bounded) number of players, using suitable *gadgets*.

  Some key gadgets derived from [Bubelis'79]'s construction.

  (Alternatively, the gadgets in [Gol-Pap'06],[Das-Gol-Pap'06] can also be used.)
- Reduce to 3-players: again uses [Bubelis '79].

Another FIXP-complete problem: Price Equilibria

- An idealized *exchange economy* with n agents and m commodities.
- For a given price vector, p, each agent l has an excess demand function $g_i^l(p)$. The total excess demand for commodity i is $g_i(p) = \sum_l g_i^l(p)$. Excess demands are continuous and satisfy economically justified axioms:
  - (Homogeneous): For all $\alpha > 0$, $p \geq 0$, $g_i^l(\alpha p) = g_i^l(p)$.
  - (Walras's law): $\sum_i p_i g_i(p) = 0$.
- *Price Equilibrium:* prices $p^* \geq 0$ such that $g_i(p^*) \leq 0$ for all i ($= 0$ if $p_i^* > 0$).
- **Fact:** Every exchange economy has a price equilibrium. (Proof via Brouwer.)
- **Proposition** *Computing Price Equilibria in exchange economies where excess demands are given by algebraic circuits over* $\{+, *, -, /, \max, \min\}$ *is FIXP-complete.* (Follows from Uzawa (1962).)

A new characterization of PPAD

Let **linear-FIXP** denote the subclass of FIXP where the algebraic circuits are restricted to basis $\{+, \max\}$ and multiplication by rational constants only.

Theorem *The following are all equivalent:*

1. *PPAD*
2. *linear-FIXP*
3. *exact fixed point problems for "polynomial piecewise-linear functions"*

(**Corollary:** `Simple-Stoch-Games` (and `Parity Games`) are in PPAD.)

sketch proof that PPAD $\leq$ linear-FIXP

Computing a 2-player NE (exactly) is PPAD-complete, so we only need to give a reduction from two player NE to linear-FIXP.

Nash's functions F_Γ are already non-linear even for 2 players.

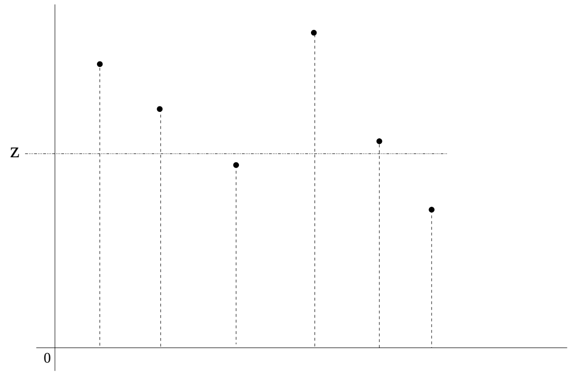
Is there a different, $\{+, \max\}$ function for 2-player NEs??

Yes!

[Gul-Pearce-Staccetti'93] describe a fixed point approach for NEs.

By examining carefully what they do, one can derive the follow function for NEs:

1. First, let $x'_{i,j} := x_{i,j} + U_i(x_{-i}; j)$.



2. Second, "*project*" the vector x'_i onto the simplex Δ_{m_i}, for every player i.

Fact: The fixed points of this function are the NEs.

Can "projection" be computed with a linear-FIXP function?
Yes, ... with the help of *sorting networks*.

From this revised function for n-player NEs we also obtain:

Theorem: Basis $\{+, *, \max\}$ (and rational constants) suffices to capture **FIXP**.

Shapley's Stochastic Games

2-player, zero-sum, imperfect information, discounted stochastic games.

1. finite state space, finite move alphabet.
2. Starting in a given state, at each round both players (independently), choose a move, or a probability distribution on moves. Their joint move determines a probability distribution on the next state, and a reward to player 1.
3. The rewards after each round are discounted by given factor $0 < \beta < 1$, and the total discounted reward to player 1 is sum $\sum_i \beta^i r_i$.

The *value* of Shapley's games (which can be irrational) can be characterized by fixed point equations, $\mathbf{x} = P(\mathbf{x})$, where $P(\mathbf{x})$ is a *contraction* map.

There is a unique *Banach* fixed point (which can be irrational), which yields the game value starting at each state.

Theorem *For Shapley's stochastic games:*

1. *Computing the game value is in FIXP.*

2. *The (strong) approximation problem for the game value is in PPAD.*

3. *The decision problem (is the game value $\geq r$?) is SqrtSum-hard.*

Proof of part (2.): $P(\mathbf{x})$ is a "fast enough" contraction mapping. For such mappings, Weak ϵ-fixed points are "close enough" to the actual Banach fixed point. $P(\mathbf{x})$ is a Brouwer function on a "not too big" domain.
Thus: apply Scarf's algorithm to $P(\mathbf{x})$. ■

Note: this implies Condon's Simple Stochasic Games are also in PPAD.

multi-type Branching Processes

Branching processes, originally studied in the 19th century by Galton and Watson.

Kolmogorov (1947) defined and studied *multi-type Branching Processes* (mt-BPs) with Sevastyanov and others.
Huge literature in probability theory, population genetics, and many other areas.

1. A population of *individuals*. Each individual has one of a fix set of *types*.

2. In each generation, every individual of a given type "gives birth" to a number of (a multi-set of) individuals of different types, according to a probability distribution on multi-sets, based on its type.

Question: Starting from one entity of a given type, will the population eventually go extinct with probability $\geq 1/2$?

(Whether it will *almost surely* go extinct is decidable in P-time ([EY05]).)

The extinction problem for mt-BPs is in FIXP

The extinction probabilities are the *Least Fixed Point* (LFP) solution of a *monotone* system of nonlinear polynomial equations, $\mathbf{x} = P(\mathbf{x})$.

(The LFP exists, by Tarski's (Tarski-Knaster) fixed point theorem.)

The LFP can be irrational, and the associated decision problems are SqrtSum-hard and PosSLP-hard ([EY05,EY07]).

The LFP can be "*isolated*" as the unique fixed point of FIXP function.

Theorem: *The mt-BP extinction problem is in FIXP.*

Note: mt-BP extinction $\equiv$ 1-exit *Recursive Markov Chain* termination

Theorem *Any non-trivial approximation of the general multi-exit RMC termination problem is SqrtSum-hard and PosSLP-hard.*

Conclusions

A very rich landscape with many, many, open questions:

- Can strong approximation of NEs be done in anything better than **PSPACE**?
- Is strong approximation of NEs hard for a standard complexity class like **NP**? (Not likely to be easy. Would imply the "*rational fragment of*" the BSS class $\mathbf{NP}_{\mathbb{R}}$ contains both NP and coNP. That's an open problem.)
- A basic practical question: Is there any algorithm that, given a game & $\epsilon > 0$:
  1. is guarranteed to output a point x within distance ϵ of some actual NE, and
  2. performs "reasonably well" in practice?

K. Etessami and M. Yannakakis, "On the complexity of Nash Equilibria and Other Fixed Points", FOCS'07.
(See full version of paper at: `http://homepages.inf.ed.ac.uk/kousha`)