A remark about algebraicity in complete partial orders

Draft

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Abstract

I prove a characterization theorem for algebraic bounded complete cpos similar to that for algebraic lattices.

It is well-known that a lattice is algebraic iff it isomorphic to a lattice of subalgebras of an algebra. Algebraicity plays the central role in denotational semantics for programming languages, but the structures used there are not exactly algebraic lattices — they are complete algebraic partial orders. In this note I shall characterize such posets as posets of certain subalgebras of partial algebras.

Let me recall the definitions. A poset is called *complete* (and is usually abbreviated as a *cpo*) if it contains least upper bounds, or suprema, of directed subsets. I shall use $\sqcup$ for supremum. An element x of D is called *compact* if $x \leq \sqcup X$, where $X \subseteq D$ is directed, implies $x \leq x'$ for some $x' \in X$. A cpo is called *algebraic* if, for any $x \in D$, the set of compact elements below x is directed and its supremum equals x.

A cpo D is *bounded complete* if supremum of $X \subseteq D$, denoted by $\sqcup X$ as well, exists whenever X is bounded above in D, i.e. there is $a \in D$ such that $a \geq x$ for all $x \in X$. I shall use a more convenient notation $a_1 \vee \ldots \vee a_n$ instead of $\sqcup\{a_1, \ldots, a_n\}$. An element x of a bounded complete cpo D is compact if, whenever $\sqcup X$ exists and $x \leq \sqcup X$, $x \leq \sqcup X'$ where $X' \subseteq X$ is finite. In a bounded complete cpo the set of compact elements below any element is always directed; therefore, a bounded complete cpo is algebraic if any element is the supremum of all compact elements below it. Algebraic bounded complete cpos are also called *Scott-domain*. Equivalently, a Scott-domain is an algebraic cpo which is a complete meet-semilattice[1].

Algebraic lattices are a particular example of Scott-domains; namely, they are Scott-domains with top element. The well-known characterization of algebraic lattices as lattices of subalgebras gives rise to a natural question: can a similar characterization be obtained for Scott-domains? The answer is yes; and a couple of definitions is needed before we formulate the result of this paper.

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[1]I must notice that program semantics people usually require that the set of compact elements be countable and this requirement is forced if we need certain computability conditions. No cardinality restriction on Scott-domains is imposed in this paper.

Let $\langle A, \Omega \rangle$ be a partial algebra with carrier A and signature Ω, and let Ω_n denote the set of n-ary operations in Ω. A (partial) subalgebra of $\langle A, \Omega \rangle$ is $B \subseteq A$ such that, for any n, $\omega \in \Omega_n$ and $x_1, \ldots, x_n \in B$, $\omega(x_1, \ldots, x_n) \in B$ if it is defined. It is also known that partial subalgebras of a partial algebra form an algebraic lattice.

I shall call $B \subseteq A$ a *total subalgebra* if, for any n, $\omega \in \Omega_n$ and $x_1, \ldots, x_n \in B$, $\omega(x_1, \ldots, x_n)$ exists and $\omega(x_1, \ldots, x_n) \in B$. The set of all total subalgebras of $\langle A, \Omega \rangle$ under inclusion ordering is denoted by $\mathrm{TSub}A$.

Theorem *Let D be a poset. Then D is a Scott-domain iff there is a partial algebra $\langle A, \Omega \rangle$ such that D is isomorphic to* $\mathrm{TSub}A$.

Proof: Prove that $\mathrm{TSub}A$ is a Scott-domain first. Obviously $\mathrm{TSub}A$ is closed under arbitrary intersections and, therefore, it is a complete meet-semilattice. If $(A_i)_{i \in I}$ is a directed family of total subalgebras of A, let $A' = \bigcup_{i \in I} A_i$. Then it is easy to check that A' is a total subalgebra again. Hence, $\mathrm{TSub}A$ is a cpo. Let $\mathsf{supp}(A) = \{x \in A \mid \exists$ a total subalgebra $B \subseteq A$ such that $x \in B\}$. Let A_x be the minimal total subalgebra containing $x \in \mathsf{supp}(A)$ (which exists since $\mathrm{TSub}A$ is a complete meet-semilattice). If, for a directed family $(A_i)_{i \in I}$, $A_x \subseteq \bigcup_{i \in I} A_i$, then $x \in \bigcup_{i \in I} A_i$ and therefore there is $j \in I$ such that $x \in A_j$. Then, by the definition of A_x, $A_x \subseteq A_j$. Therefore, each A_x is compact in $\mathrm{TSub}A$. Since for any $B \in \mathrm{TSub}A$, $B \subseteq \mathsf{supp}(A)$, B is the supremum of all A_x where $x \in B$ in $\mathrm{TSub}A$. Hence, compact elements form a basis of $\mathrm{TSub}A$ and $\mathrm{TSub}A$ is algebraic.

Conversely, let D be a Scott-domain. Denote the set of its compact elements by $\mathbf{K}D$. We are going to define an algebra whose carrier is $A = \mathbf{K}D$ and partial operations are defined as follows. An operation $\omega \in \Omega_n$ is given by: for any $a_1, \ldots, a_n \in \mathbf{K}D$, if $\{a_1, \ldots, a_n\}$ is not bounded above, then $\omega(a_1, \ldots, a_n)$ is undefined; if this set is bounded above, i.e. $a_1 \vee \ldots \vee a_n$ exists, then $\omega(a_1, \ldots, a_n) = b$ where $b \leq a_1 \vee \ldots \vee a_n$ and $b \in \mathbf{K}D$. We define just enough operations so that for any n, $b \leq a_1 \vee \ldots \vee a_n$, $b, a_1, \ldots, a_n \in \mathbf{K}D$, there exists $\omega \in \Omega_n$ such that $b = \omega(a_1, \ldots, a_n)$.

To finish the proof, we must show that $B \in \mathrm{TSub}A$ iff there exists $x \in D$ such that $B = \downarrow x \cap \mathbf{K}D$, where $\downarrow x$ is the principal ideal of x. The *'if'* part follows immediately from the definition of Ω. To prove the *'only if'* part, let $B \in \mathrm{TSub}A$. Then for any $a_1, \ldots, a_n \in B$, $a_1 \vee \ldots \vee a_n$ exists (otherwise there would be an n-ary operation undefined on $a_1, \ldots, a_n$). Therefore, $M = \{a_1 \vee \ldots \vee a_n \mid a_1, \ldots, a_n \in B\}$ is a directed set, and we can define x as $\sqcup M = \sqcup B$. By the way x was defined, $B \subseteq \downarrow x \cap \mathbf{K}D$. If $b \in \downarrow x \cap \mathbf{K}D$, then $b \leq \sqcup B$, and, since b is compact, $b \leq a_1 \vee \ldots \vee a_n$ for some $a_1, \ldots, a_n \in B$. Then there is an operation $\omega \in \Omega_n$ such that $b = \omega(a_1, \ldots, a_n)$, i.e. $b \in B$. This proves the reverse inclusion and finishes the proof of the theorem. □

A similar characterization theorem can be proven if partiality is removed from the signature to the subsets which are allowed to play the role of subalgebras. To be more precise, let me define $\mathcal{F}$ as a family of subsets of a set A which is downward closed (i.e. $B \in \mathcal{F}$ and $C \subseteq B$ imply $C \in \mathcal{F}$) and complete as a partial order under inclusion. Such a family is sometimes called a *qualitative domain.* Given a signature Ω, define $\mathrm{Sub}_{\mathcal{F}}A$ as the poset of subalgebras of $\langle A, \Omega \rangle$ which happen to be in $\mathcal{F}$.

Corollary *A poset D is a Scott-domain iff there exist an algebra $\langle A, \Omega \rangle$ and a qualitative domain $\mathcal{F}$ of subsets of A such that D is isomorphic to* $\mathrm{Sub}_{\mathcal{F}}A$.

Proof is essentially the same as the one given above. A is taken to be $\mathbf{K}D$, and $\mathcal{F}$ is the family of subsets of $\mathbf{K}D$ bounded above. □

Arbitrary cpos can also be viewed as posets of certain subalgebras of "nondeterministic" algebras, i.e. algebras with operations whose results are not uniquely defined even if they are defined.

More precisely, define a *partial nondeterministic algebra* as $\langle A, \Omega \rangle$ where each $\omega \in \Omega_n$ is partial function from A^n to 2^A. We say that $B \subseteq A$ is a total subalgebra of A if, for any $a_1, \ldots, a_n \in B$ and $\omega \in \Omega_n$, $\omega(a_1, \ldots, a_n)$ is defined and $\omega(a_1, \ldots, a_n) \cap B \neq \emptyset$. The poset of total subalgebras under inclusion ordering will be denoted by $\mathrm{TSub}A$ again. An example can be given showing that $\mathrm{TSub}A$ is not necessarily algebraic if A has "nondeterministic" operations. However,

Proposition 1 *Let D be an algebraic cpo. Then there is a partial nondeterministic algebra $\langle A, \Omega \rangle$ such that D is isomorphic to* $\mathrm{TSub}A$.

Proof: A bunch of deterministic operations are defined as in the proof of the theorem with only one exception: instead of saying $b \leq a_1 \vee \ldots \vee a_n$ we say that b is less than any upper bound of $\{a_1, \ldots, a_n\}$. The only n-ary nondeterministic operation ω is undefined for n-tuples of compact elements which are unbounded; if $\{a_1, \ldots, a_n\}$ is bounded above, then $\omega(a_1, \ldots, a_n) = \{b \in \mathbf{K}D \mid b \geq a_1, \ldots, a_n\}$. Notice that $\omega(a_1, \ldots, a_n)$ is nonempty. It can be easily shown that for such a signature Ω, $\mathrm{TSub}\langle \mathbf{K}D, \Omega \rangle$ is isomorphic to D. $\square$

This construction sheds some light on well-known condition "M". The condition M says that, for any finite bounded set X of compact elements, the set of its minimal upper bounds if finite, and each upper bound of X is greater than some minimal upper bound. Let 2^A_{fin} stand for the family of all finite subsets of A.

Proposition 2 *Let D be an algebraic cpo satisfying "M". Then there is a partial nondeterministic algebra $\langle A, \Omega \rangle$ in which any n-ary operation is of form $A^n \to 2^A_{fin}$ such that D is isomorphic to* $\mathrm{TSub}A$.

Proof is the same as above, but we take $\omega(a_1, \ldots, a_n)$ to be the set of minimal upper bounds if it exists and undefined otherwise. $\square$

References

1. Grätzer, Universal Algebra.
2. Qualitative domains.
3. Condition "M".