The Lattice of Subsemilattices of a Semilattice

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This note makes two observations about lattices of subsemilattices. First, we establish relationship between direct decompositions of such lattices and ordinal sum decompositions of semilattices. Then we give a characterization of the subsemilattice-lattices.

Let us recall some terminology. L will always stand for a semilattice, whose operation will be denoted by $\circ$. The ordering on L is given by letting $l_1 \leq l_2$ iff $l_1 \circ l_2 = l_2$, i.e. L is always a join-semilattice. Subsemilattices of L, ordered by inclusion, form a subsemilattice-lattice denoted by $Sub\, L$. In $Sub\, L$ the meet operation is intersection, and the join operation is defined as follows: $L_1 \vee L_2 = L_1 \cup L_2 \cup \{l_1 \circ l_2 \mid l_1 \in L_1, l_2 \in L_2\}$. An element a of an arbitrary lattice $\mathcal{L}$ is called *neutral* if $m(a, x, y) = M(a, x, y)$ for all $x, y \in \mathcal{L}$, where $m(a, x, y) = (a \wedge x) \vee (a \wedge y) \vee (x \wedge y)$ and $M(a, x, y) = (a \vee x) \wedge (a \vee y) \wedge (x \vee y)$. Notice that $m(a, x, y) \leq M(a, x, y)$ holds in any lattice.

Lemma 1 *Let L be a semilattice and L_0 its subsemilattice. Then L_0 is a neutral element of $Sub\, L$ iff $L - L_0$ is a subsemilattice of L and every element of L_0 is comparable with every element of $L - L_0$.*

Proof. Let L_0 be a subsemilattice of L such that $L - L_0$ is a subsemilattice of L as well and every element of L_0 is comparable with every element of $L - L_0$. We must prove that, for any $L_1, L_2 \in Sub\, L$, $M(L_0, L_1, L_2) \subseteq m(L_0, L_1, L_2)$. Let $x \in M(L_0, L_1, L_2)$. Since $L_0 \vee L_i = L_0 \cup L_i$, $i = 1, 2$, there are 12 cases, but only one of them is nontrivial: $x \in L_0$ and $x = l_1 \circ l_2$, where $l_1 \in L_1, l_2 \in L_2$. If l_1 and l_2 are comparable, then either $x \in L_1$ or $x \in L_2$; hence $x \in m(L_0, L_1, L_2)$. If l_1 and l_2 are not comparable, then $l_1, l_2 \in L_0$ and $x \in (L_0 \wedge L_1) \vee (L_0 \wedge L_2) \subseteq$

*While working on the final version, this author was supported by NSF Grants IRI-86-10617 and CCR-90-57570 and ONR Grant NOOO14-88-K0634.

$m(L_0, L_1, L_2)$. Conversely, if $L_0 \in Sub\,L$ and $L - L_0$ is not a subsemilattice of L, then there exist $l_1, l_2 \notin L_0$ such that $l_1 \circ l_2 \in L_0$. But then $m(L_0, \{l_1\}, \{l_2\}) \neq M(L_0, \{l_1\}, \{l_2\})$. If $L - L_0$ is a subsemilattice of L and there exist incomparable $l_1 \in L_0, l_2 \notin L_0$ and $l = l_1 \circ l_2$, then $m(L_0, L - L_0, \{l_2\}) \neq M(L_0, L - L_0, \{l_2\})$ if $l \notin L_0$ and $m(L_0, \{l_1\}, \{l_2\}) \neq M(L_0, \{l_1\}, \{l_2\})$ if $l \in L_0$. Hence, L_0 is not neutral. □

Lemma 2 *$Sub\,L \simeq \mathcal{L}_1 \times \mathcal{L}_2$ iff there exists a neutral element L_0 of $Sub\,L$ such that $\mathcal{L}_1 \simeq Sub\,L_0$ and $\mathcal{L}_2 \simeq Sub\,L - L_0$.*

Proof. By theorem 1 of [2, p. 152], the direct decompositions of $Sub\,L$ into two factors are of form $Sub\,L \simeq (L_0] \times [L_0)$, where L_0 is neutral. By lemma 1, $\varphi : Sub\,L - L_0 \to [L_0)$ defined by $\varphi(L') = L' \cup L_0$ is a lattice isomorphism if L_0 is neutral. The lemma follows now from the fact that $Sub\,L_0 \simeq (L_0]$. □

Corollary 1 *An arbitrary semilattice L can not be represented as an ordinal sum of its proper subsemilattices iff $Sub\,L$ is directly indecomposable.* □

Corollary 2 *If L is finite, then $Sub\,L$ is directly indecomposable iff it is subdirectly irreducible.*

Proof. One direction is obvious. To prove that a directly indecomposable $Sub\,L$ is subdirectly irreducible, assume that $|L| \geq 2$, since $Sub\,L$ for a one-element L is a two-element chain and, therefore, subdirectly irreducible. Let $\mathbf{1}$ be the greatest element of L. We will show that $\Theta(\emptyset, \{\mathbf{1}\})$ is a unique atom of the congruence lattice of $Sub\,L$. Since one-element subsemilattices are exactly the atoms of $Sub\,L$, it is enough to show that $\Theta(\emptyset, \{\mathbf{1}\}) \leq \Theta(\emptyset, \{l\})$ for each $l \in L, l \neq \mathbf{1}$ or, equivalently, that $\{\mathbf{1}\}/\emptyset \approx_\omega \{l\}/\emptyset$. Notice that if $l_1 \circ l_2 = l$ in L, then $\{l\}/\emptyset \sim_\omega \{l_1, l_2, l\}/\{l_2\} \sim_\omega \{l_1\}/\emptyset$ in $Sub\,L$.

Since $Sub\,L$ is directly indecomposable, by corollary 1 for any element $l \in L, l \neq \mathbf{1}$, there exists $l' \in L$ incomparable with l, i.e. $l \circ l' > l$. Since L is finite, for any $l \neq \mathbf{1}$ there is a finite sequence $l_0, l_1, \ldots, l_{2n}$, where $l_0 = l, l_{2n} = \mathbf{1}$, l_{2i} and l_{2i+1} are incomparable and $l_{2i+2} = l_{2i} \circ l_{2i+1}$, $i = 0, \ldots, n-1$. The existence of such a sequence and the observation made above immediately imply $\{\mathbf{1}\}/\emptyset \approx_\omega \{l\}/\emptyset$. □

Notice that any neutral element of $Sub\,L$ is complemented. Neutral complemented elements of any lattice $\mathcal{L}$ form a Boolean sublattice of $\mathcal{L}$ denoted by $Cen(\mathcal{L})$ [2]. It follows from lemma 1 that intersection of an arbitrary family of neutral elements of $Sub\,L$ is neutral. Hence, $Cen(Sub\,L)$ is a complete lattice. Moreover, intersection of all neutral elements containing $l \in L$ is an atom of $Cen(Sub\,L)$. Therefore, $Cen(Sub\,L)$ is an atomic Boolean lattice whose atoms are exactly ordinally indecomposable subsemilattices of L. From this we conclude

Theorem 1 *Let L be an arbitrary semilattice. Then $Sub\,L$ can be represented as a direct product of directly indecomposable lattices, $Sub\,L \simeq \prod_{i \in I} Sub\,L_i$, where $L = \oplus_{i \in I} L_i$ is a representation of L as an ordinal sum of ordinally indecomposable subsemilattices.* □

In the finite case the structure of $Cen(Sub\,L)$ allows us to list all the direct decompositions of $Sub\,L$. If $L = \oplus_{i\in I}L_i$, where each L_i is ordinally indecomposable and $Sub\,L \simeq \mathcal{L}_1 \times \ldots \mathcal{L}_m$, then there exist disjoint sets $I_1, \ldots, I_m \subseteq I$ such that $I_1 \cup \ldots \cup I_m = I$, $L'_j = \oplus_{i\in I_j} L_i$ and $\mathcal{L}_j \simeq Sub\,L'_j$ for $j = 1, \ldots, m$.

We conclude the paper by characterizing the subsemilattice-lattices. An atomistic lattice $\mathcal{L}$ is called *biatomic* [1] if for any two non-zero $x, y \in \mathcal{L}$ and an atom $z \leq x \vee y$ there exist atoms $x' \leq x, y' \leq y$ such that $z \leq x' \vee y'$. We say that a biatomic lattice $\mathcal{L}$ satisfies property (S_n) if for any ideal V generated by n atoms $a_1, \ldots, a_n \in \mathcal{L}$ there exists a finite semilattice L_V such that $V \simeq Sub\,L_V$, and the natural embedding of ideals $V \to W$ induces the embedding of semilattices $L_V \to L_W$.

Theorem 2 *A lattice $\mathcal{L}$ is isomorphic to $Sub\,L$ for some semilattice L iff it is algebraic, biatomic and satisfies (S_3).*

Proof. The *'only if'* part is obvious. To prove the *'if'* part, denote the set of atoms of $\mathcal{L}$ by $A(\mathcal{L})$ and the set of atoms under $x \in \mathcal{L}$ by $A(x)$. Notice that (S_3) implies that $(*)$ for every $X \subseteq A(\mathcal{L})$ with $|X| \leq 3$ there exists a semilattice operation $\circ_X$ on $A(\bigvee X)$ such that $(\bigvee X] \simeq Sub\,\langle A(\bigvee X), \circ_X\rangle$ and $\circ_Y = \circ_X|_{A(\bigvee Y)}$ for every $Y \subseteq A(\bigvee X)$ with $|Y| \leq 3$.

Define a binary operation $\circ$ on $A(\mathcal{L})$ by $a_1 \circ a_2 = a_1 \circ_{\{a_1,a_2\}} a_2$. Clearly, $\circ$ is idempotent and commutative. That $\circ$ is associative follows from $(*)$. Thus, $\circ$ is a semilattice operation on $A(\mathcal{L})$. Define $\varphi : \mathcal{L} \to Sub\,\langle A(\mathcal{L}), \circ\rangle$ by $\varphi(y) = \{x \in A(\mathcal{L}) \mid x \leq y\}$. That φ is well-defined follows from $(*)$. The remaining properties of $\mathcal{L}$ guarantee that φ is an isomorphism. Thus, $\mathcal{L} \simeq Sub\,\langle A(\mathcal{L}), \circ\rangle$. $\square$

Remark. Neutral elements of a lattice $Sub\,L$ were characterized in lemma 1. One can easily check that a weaker condition characterizes distributive and standard elements. In fact, L_0 is a distributive element of $Sub\,L$ iff it is standard iff for all $l_1 \in L_0, l_2 \notin L_0$ either $l_1 \leq l_2$ or $l_1 \circ l_2 \in L_0$.

Acknowledgements: We would like to thank E.T. Schmidt and W. Dziobiak for their helpful comments.

References

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