# Direct Decompositions of Atomistic Algebraic Lattices

Leonid Libkin*

*Department of Computer and Information Science*
*University of Pennsylvania, Philadelphia, PA 19104-6389, USA*

## 1 Introduction

It is well-known that while any algebra is a subdirect product of subdirectly irreducible algebras, the analog of this result is not generally true for direct product. However, it is true for some classes of algebras, for example, for lattices of finite length. In this paper we prove the existence of such decompositions for two classes of lattices. First, we prove

**Theorem 1** *Every atomistic algebraic lattice is a direct product of directly indecomposable (atomistic algebraic) lattices.*

(A lattice is called *atomistic* if every element is the join of atoms below it). The proof is given by describing the structure of neutral complemented elements of such lattices. Then we show how theorem 1 can be generalized. An element $x$ of a complete lattice $L$ is called *strictly join-irreducible* if $x = \bigvee X$ implies $x \in X$ for an arbitrary $X \subseteq L$. Borrowing the terminology from [8], we call a lattice in which every element is the join of strictly join-irreducible elements below it a $V_1-$*lattice*. Our next result is:

**Theorem 2** *Every algebraic $V_1$-lattice is a direct product of directly indecomposable (algebraic $V_1$-) lattices.*

While theorem 1 can be viewed as a corollary of theorem 2, we prefer to prove theorem 1 first and then outline the proof of theorem 2. The reason for this order of presentation is twofold. Firstly, the proof of theorem 1 demonstrates the nice structure of distributive, standard and neutral elements in atomistic algebraic lattices. Such concise characterizations for those elements can not be obtained for $V_1$-lattices although the characterizations that will be obtained are sufficient to prove theorem 2. Secondly, theorem 1 has a clear interpretation when it is reformulated in terms of subalgebra-lattices of idempotent algebras.

The rest of the paper is organized in two sections. Section 2 contains the proof of theorem 1 and its applications to subalgebra-lattices of idempotent algebras. In Section 3 an outline of the proof of theorem 2 is given.

*Supported in part by AT&T Doctoral Fellowship and NSF Grant IRI-90-04137.

## 2 Proof of theorem 1

Recall the definitions of distributive, standard and neutral elements [5]. An element $a \in L$ is called *distributive (standard* or *neutral)* if for any $x, y \in L$ (1) ((2) or (3)) holds:

(1) $$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$$

(2) $$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$$

(3) $$(x \wedge a) \vee (y \wedge a) \vee (x \wedge y) = (x \vee a) \wedge (y \vee a) \wedge (x \vee y).$$

Given a lattice $L$, its center $Cen(L)$ is defined to be the set of neutral complemented elements. $Cen(L)$ is a sublattice of $L$. Moreover, it is Boolean.

To prove theorem 1 it suffices to show that $Cen(L)$ is a complete atomistic Boolean lattice. Then, if $A_c$ is the set of atoms of $Cen(L)$, $Cen(L) \cong \mathbf{2}^A$ and $L \cong \prod_{a \in A}(a]$ is a representation of $L$ as a direct product of directly indecomposable lattices. This suggests the following strategy: give a characterization of neutral complemented elements of $L$ which will make it easy to prove that $Cen(L)$ is complete and atomistic. This is done in four steps, represented by lemmas 1–4.

We need some new terminology before we proceed to prove the lemmas. If $p_1, \ldots, p_n \in L$, then $p^i = \bigvee_{j \neq i} p_j$. An inequality $r \leq p_1 \vee \ldots \vee p_n$ is called *minimal* if $r \not\leq p^i$ for any $i$. An element $x \in L$ is called a *face element* if $r \leq x$ and $r \leq p_1 \vee \ldots \vee p_n$ where $r$ and all $p_i$'s are atoms and the inequality is minimal imply $p_1 \vee \ldots \vee p_n \leq x$ (the name is suggested by a lattice theoretic characterization of faces of polytopes as face elements of the lattices of convex subsets of polytopes [1]).

Throughout this section, $L$ is an atomistic algebraic lattice and $A$ is the set of its atoms. $A(x)$ is $A \cap (x]$.

**Lemma 1** *Given $x \in L$, the following are equivalent:*
*1) $x$ is distributive;*
*2) $x$ is standard;*
*3) For any $y \in L$, $A(x \vee y) = A(x) \cup A(y)$.*

*Proof:* 1) $\Rightarrow$ 3). Since $A(x \vee y) \supseteq A(x) \cup A(y)$, we have to prove the reverse inequality. Let $p$ be an atom and $p \leq x \vee y$. Since $p$ is compact, $p \leq a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where all $a_i$'s and $b_j$'s are atoms, $a_i \leq x, b_j \leq y$. If all $b_j$'s are below $x$, then $p \leq x$ and we are done. So, assume without loss of generality that no $b_j$ is below $x$. Let $b = b_1 \vee \ldots \vee b_m$. Then $(x \vee p) \wedge (x \vee b) \geq p$. Since $x$ is distributive, $x \vee (p \wedge b) \geq p$ too. If $p \leq b$, then $p \leq y$, and if $p \not\leq b$, then $p \leq x$. Hence, $p \in A(x) \cup A(y)$. The implication 3) $\Rightarrow$ 2) is straightforward and 2) $\Rightarrow$ 1) is known [5]. □

**Lemma 2** *If $x \in L$ is a neutral element, then it is complemented and $A(\overline{x}) = A - A(x)$.*

*Proof:* Let $x$ be neutral, $A' = A - A(x)$ and $\overline{x} = \bigvee A'$. It is enough to show that $x \wedge \overline{x} = 0$. Suppose that there is an atom $p \leq x \wedge \overline{x}$. Since $p$ is compact, $p \leq a_1 \vee \ldots \vee a_n$ where all $a_i$'s are atoms in

$A'$. Assume without loss of generality that the inequality is minimal. Clearly, $n > 1$ as $p \le x$. Then $a^1 \ne 0$ and $(x \vee a_1) \wedge (x \vee a^1) \wedge (a_1 \vee a^1) \ge p$. By neutrality, $(x \wedge a_1) \vee (x \wedge a^1) \vee (a^1 \wedge a_1) \ge p$ which, by minimality, rewrites to $x \wedge a^1 \ge p$ as $a_1$ cannot be below $a^1$. But then $p \le a^1$ and this contradicts minimality. Hence, $x \wedge \overline{x} = 0$. □

**Corollary 1** *$Cen(L)$ is the sublattice of neutral elements.* □

**Lemma 3** *$x \in Cen(L)$ iff it is a standard face element.*

*Proof:* Let $x$ be in $Cen(L)$. Then $x$ is standard and we must show it is a face element. Let $r \le x, r \le p_1 \vee \ldots \vee p_n$, where the inequality is minimal. If $n = 1$, we are done. If $n > 1$, assume that $p_i \not\le x$. Then $(x \vee p_i) \wedge (x \vee p^i) \wedge (p_i \vee p^i) \ge r$; hence $(x \wedge p_i) \vee (x \vee p^i) \vee (p_i \wedge p^i) \ge r$ by neutrality. By minimality of the inequality $r \le p_1 \vee \ldots \vee p_n$ this rewrites to $x \wedge p^i \ge r$, i.e. $r \le p^i$ and this contradicts minimality. Hence, $p_i \le x$. Since $i$ was chosen arbitrarily, all $p_i$'s are below $x$ and $x$ is a face element.

Conversely, let $x$ be face and standard. We must show that $x$ is neutral, i.e. for any atom $p$, $p \le (x \wedge a) \vee (x \wedge b) \vee (a \wedge b)$ whenever $p \le (x \vee a) \wedge (x \vee b) \wedge (a \vee b)$. By lemma 1, $p \in (A(x) \cup A(a)) \cap (A(x) \cup A(b)) = A(x) \cup (A(a) \cap A(b))$ and $p \le a \vee b$. If $p \in A(a) \cap A(b)$, then $p \le a \wedge b$ and we are done. Let $p \le x$. Since $p \le a \vee b$, by compactness of $p$: $p \le a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i$'s are atoms under $a$, $b_j$'s are atoms under $b$ and the inequality is minimal. Since $p \le x$ and $x$ is a face element, all $a_i$'s and $b_j$'s are under $x$. But then $p \le (x \wedge a) \vee (x \wedge b)$ finishes the proof of neutrality. □

**Lemma 4** *$x \in Cen(L)$ iff it satisfies the following conditions:*
*(1) $x$ is complemented and for its complement $\overline{x}$ it holds: $\overline{x} = \bigvee(A - A(x))$;*
*(2) if $p \in A$ and $p \le a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i \in A(x), b_j \in A - A(x)$, $i = 1, ..., n, j = 1, ..., m$ then either $p \le a_1 \vee \ldots \vee a_n$ or $p \le b_1 \vee \ldots \vee b_m$.*

*Proof:* Let $x \in Cen(L)$. We have already shown (1) (lemma 2) and only (2) remains to be proved. There are two cases: $p \le x$ or $p \le \overline{x}$ as $A = A(x) \cup A(\overline{x})$. Let $p \le x$. ¿From the set $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ remove all atoms such that $p$ is below the join of the rest and repeat this operation until no new atom can be removed. This results in a minimal inequality $p \le a_{i_1} \vee \ldots \vee a_{i_k} \vee b_{j_1} \vee \ldots \vee b_{j_l}$. By lemma 3, all these atoms must be under $x$; hence $l = 0$ and $p \le a_{i_1} \vee \ldots \vee a_{i_k} \le a_1 \vee \ldots \vee a_n$. If $p \le \overline{x}$, let $b = b_1 \vee \ldots \vee b_m$. Then $(x \vee p) \wedge (x \vee b) \wedge (b \vee p) \ge p$; hence $(x \wedge p) \vee (x \wedge b) \vee (b \wedge p) = (x \wedge b) \vee (b \wedge p) \ge p$. If $p \le b$, then we are done. If $p \not\le b$, then $p \le x \wedge b = 0$, a contradiction. Hence, $p \le b$.

Conversely, let $x$ satisfy (1) and (2). Prove that it is standard. Let $p \le x \vee y$, where $p$ is an atom. Then $p \le a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i$'s are atoms under $x$ and $b_j$'s are atoms under $y$. Without loss of generality, assume $b_j \not\le x$ for all $j$. Then by (2) $p \le a_1 \vee \ldots \vee a_n \le x$ or $p \le b_1 \vee \ldots \vee b_m \le y$, i.e. by lemma 1 $x$ is standard. To prove that $x$ is a face element, let $r \le p_1 \vee \ldots \vee p_n$ be a minimal inequality, where $r \le x$. If all $p_i$'s are below $x$, then we are done. Since $p \le x$, there are some $p_i$'s under $x$. Let $p_1, \ldots, p_k$ be below $x$ and $p_{k+1}, \ldots, p_n$ below $\overline{x}$. Then by (2) $p \le p_1 \vee \ldots \vee p_k$ and by minimality $k = n$, i.e. all $p_i$'s are below $x$. This proves that $x$ is a face element and by lemma 3 $x$ is in $Cen(L)$. □

Lemma 4 gives us the sought characterization. Now we can prove

**Lemma 5** *$Cen(L)$ is a complete sublattice.*

*Proof:* Let $x_i \in Cen(L)$, where $i \in I$ and $I$ is an arbitrary set of indices. Let $X_i = A(x_i)$ and $x = \bigwedge_{i\in I} x_i$. We must prove that $x \in Cen(L)$. Let $X = A(x)$. Then $X = \bigcap_{i\in I} X_i$ and $X' = A - X = \bigcup_{i\in I} X_i'$ where $X_i' = A - X_i$. Let $\overline{x} = \bigvee X'$. Prove $x \wedge \overline{x} = 0$ first. Assume that there exists an atom $p \leq x \wedge \overline{x}$. Since $p$ is compact, $p \leq a_1 \vee \ldots \vee a_n$ where all $a_i$'s are in $X'$. Therefore, there exist finitely many indices $i_1, \ldots, i_k \in I$ such that $\{a_1, \ldots, a_n\} \subseteq \bigcup_{j=1}^k X_{i_j}'$. Let $y = x_{i_1} \wedge \ldots \wedge x_{i_k}$. Since $Cen(L)$ is a sublattice of $L$, $y \in Cen(L)$ and $A(y) = \bigcap_{j=1}^k X_{i_j}$. Since $y \in Cen(L)$, it has the complement $\overline{y}$ and $A(\overline{y}) = \bigcup_{j=1}^k X_{i_j}'$. Hence $p \leq \overline{y}$ and $p \leq x \leq y$, i.e. $p \leq y \wedge \overline{y} = 0$. This contradiction shows $x \wedge \overline{x} = 0$. Now, let $p \leq a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i \in X, b_j \in X'$, $i = 1, ..., n, j = 1, ..., m$. Similarly, we can find an element $y \in Cen(L)$ such that $y$ is the meet of finitely many $x_i$'s and all $b_j$'s are in $A(\overline{y})$. Since $a_i \leq x \leq y$, we now have $p \leq a_1 \vee \ldots \vee a_n$ or $p \leq b_1 \vee \ldots \vee b_m$ by applying (2) of lemma 4 to $y$. This finishes the proof that $x \in Cen(L)$.

The proof of $x = \bigvee_{i\in I} x_i \in Cen(L)$ is similar. Thus, $Cen(L)$ is a complete sublattice of $L$. □

Finally,

**Lemma 6** *$Cen(L)$ is atomistic.*

*Proof:* Let $q \in A$. Define $c_q = \bigwedge(x \mid x \in Cen(L), x \geq q)$. By lemma 5, $c_q \in Cen(L)$. To prove that $c_q$ thus defined is an atom of $Cen(L)$, it is enough to show that for no atom $p$ is $c_p$ below $c_q$. Suppose $c_p < c_q$, i.e. $A(c_p) \subset A(c_q)$. Since $c_p, c_q \in Cen(L)$, $c_q \wedge \overline{c_p} \in Cen(L)$ and $A(c_q \wedge \overline{c_p}) = A(c_q) - A(c_p)$. If $q \in A(c_p)$, then $q \leq c_p$ and $c_q \leq c_p$; hence $q \in A(c_q) - A(c_p)$, i.e. $c_q \wedge \overline{c_p} \geq q$ and $c_q \wedge \overline{c_p} \geq c_q$. Then $\overline{c_p} \geq c_q$ and $c_p < c_q \leq \overline{c_p}$. This contradiction proves that $c_q$ is an atom in $Cen(L)$. Since $x = \bigvee_{q\in A(x)} c_q$ for any $x \in Cen(L)$, $Cen(L)$ is atomistic. □

Lemmas 5 and 6 finish the proof of theorem 1 since, if $A_c$ is the set of atoms of $Cen(L)$, $Cen(L) \cong \mathbf{2}^{A_c}$.

The characterization of distributive and neutral elements can be given in the following alternative way. Let $[\cdot]$ be the closure on $A$ defined by $[X] = A(\bigvee X)$. It is well-known that $L$ is isomorphic to the lattice of closed subsets of $A$. Lemma 1 now says that $X$ is a distributive element of the lattice of closed subsets of $A$ iff $[X \cup Y] = X \cup [Y]$ for every $Y \subseteq A$. Lemma 4 says that $X$ is a neutral element iff its complement $X' = A - X$ is also closed and $[Y] = [Y \cap X] \cup [Y \cap X']$ for any $Y \subseteq A$. (This characterization of neutral elements can also be derived from [6, theorem 2].)

It is possible to use this characterization of neutral elements to prove lemmas 5 and 6 in exactly the same way as they were proved above. The characterization given in lemmas 1 and 4 points out to a close connection with representation of convex theoretic concepts in lattices (cf. [1, 2]). It also allows us to prove two corollaries of theorem 1. Recall that an element $a \in L$ is called *codistributive* if $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$ for any $x, y \in L$.

**Corollary 2** *In an atomistic algebraic lattice an element is neutral iff it is distributive and codistributive.*

*Proof:* Any neutral element is distributive and codistributive. Let $a$ be a distributive and codistributive element of $L$. Let $A' = A - A(a)$ and $\overline{a} = \bigvee A'$. Prove that $\overline{a}$ is $a$'s complement, i.e. $\overline{a} \wedge a = 0$. Suppose $p$ is an atom under $\overline{a} \wedge a$. Then $p \leq p_1 \vee \ldots \vee p_n$ where $p_1, \ldots, p_n \in A'$ and the inequality is chosen to be minimal. Clearly, $n > 1$. By codistributivity, $p \leq a \wedge (p_1 \vee p^1) = (a \wedge p_1) \vee (a \wedge p^1) = a \wedge p^1$. Thus, $p \leq p^1$ which contradicts minimality. Prove that $\overline{a}$ is distributive too. Let $p \in A(\overline{a} \vee y)$. We must show $p \in A(\overline{a}) \cup A(y)$. If $p \in A(\overline{a})$, we are done. If $p \notin A(\overline{a})$, $p \leq a$ and $p \leq a \wedge (\overline{a} \vee y) = (a \wedge \overline{a}) \vee (a \wedge y)$, i.e. $p \in A(y)$. Thus, $\overline{a}$ is distributive. To check (2) of lemma 4, let $p \leq a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$, $a_i \leq a$, $b_j \leq \overline{a}$. Let $p \leq \overline{a}$. By distributivity of $a$, $p \leq a \vee b_1 \vee \ldots \vee b_m$ implies $p \leq b_1 \vee \ldots \vee b_m$ since $p \not\leq a$. Similarly, if $p \leq a$, then distributivity of $\overline{a}$ implies $p \leq a_1 \vee \ldots \vee a_n$. Thus, $a$ is neutral by lemma 4. $\square$

Atomistic algebraic lattices are subalgebra-lattices of idempotent algebras and one may ask how direct decompositions of such lattices can be viewed in terms of the structure of underlying algebras. The following definition which is reminiscent of lemma 4 provides us with a reformulation of theorem 1 in terms of idempotent algebras.

Let $\langle A, \Omega \rangle$ be an idempotent algebra where $A$ is a carrier and $\Omega$ is a signature. A subalgebra $B \subseteq A$ is called *splitting* if $A - B$ is also a subalgebra and, for any $b_1, ..., b_m \in B$ and $c_1, ..., c_k \in A - B$ and an $m + k$-ary operation $\omega \in \Omega$, there exists an $\Omega$-term $t$ such that $\omega(b_1, \ldots, b_m, c_1, \ldots, c_k)$ is either $t(b_1, \ldots, b_m)$ or $t(c_1, \ldots, c_k)$.

**Corollary 3** *Any idempotent algebra is the union of its minimal splitting subalgebras and its subalgebra-lattice is isomorphic to the product of subalgebra-lattices of the minimal splitting subalgebras. Moreover, the subalgebra-lattice of an idempotent algebra $\langle A, \Omega \rangle$ is directly indecomposable iff $\langle A, \Omega \rangle$ has no proper splitting subalgebras.* $\square$

In the case of lattices and semilattices splitting subalgebras are such that their elements are comparable with every element of their complements, hence the name. Corollary 3 for lattices and semilattices is known, see [4, 7].

## 3 Proof of theorem 2

The idea is similar to that of the proof of theorem 1: we give a characterization of $Cen(L)$ that makes it easy to prove that $Cen(L)$ is an atomistic Boolean lattice. Throughout this section, $L$ is an algebraic $V_1$-lattice, i.e. a lattice in which every element is the join of strictly join-irreducible elements. Let $J$ be the set of such elements; $J(x)$ is $J \cap (x]$. Clearly, any element of $J$ is compact.

The analog of lemma 1 is not that general as distributive elements are not necessarily standard in $L$.

**Lemma 7** *An element $x \in L$ is standard iff $J(x \vee y) = J(x) \cup J(y)$ for any $y \in L$.*

*Proof:* Let $x$ be standard. Let $p \in J(x \vee y)$. Then $p \leq a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i \in J(x), b_j \in J(y)$. Without loss of generality, assume $b_j \not\leq x$. Let $b = b_1 \vee \ldots \vee b_m$. Since $p \leq p \wedge (x \vee b)$ and $x$ is standard, $p \leq (p \wedge x) \vee (p \wedge b)$. By compactness of $p$, $p \leq a'_1 \vee \ldots \vee a'_k \vee b'_1 \vee \ldots \vee b'_l$ where $a'_i \in J(p \wedge x) \subseteq J(x)$ and $b'_j \in J(p \wedge b) \subseteq J(y)$. But since $a'_i, b'_j \leq p$, one has $p = a'_1 \vee \ldots \vee a'_k \vee b'_1 \vee \ldots \vee b'_l$ and $p = a'_i$ or $p = b'_j$ for some $i$ or $j$ since $p \in J$. Thus, $p \in J(x) \cup J(y)$. Conversely, if $J(x \vee y) = J(x) \cup J(y)$ for any $y$, it is almost immediate that $x$ is standard. □

Not every neutral element is complemented in a $V_1$-lattice. However,

**Lemma 8** *Let $x$ be a neutral element. Then $x$ is complemented iff for every $p \in J$, $p \not\leq x$ implies $p \wedge x = 0$.*

*Proof*: Let $x$ be complemented. Since $x$ is neutral, for any $p$: $p = (x \wedge p) \vee (\overline{x} \wedge p)$ where $\overline{x}$ is the complement of $x$. If $p \in J$, this implies $p \leq x$ or $p \leq \overline{x}$. If $p \not\leq x$, then $p \leq \overline{x}$ and $p \wedge x \leq x \wedge \overline{x} = 0$. Conversely, let the condition of lemma hold. Let $J' = J - J(x)$ and $\overline{x} = \bigvee J'$. The proof of $x \wedge \overline{x} = 0$ is essentially similar to the proof of lemma 2. Hence, $\overline{x}$ is the complement of $x$. □

**Lemma 9** *$x \in Cen(L)$ iff $x$ is standard, complemented and the following holds:*

$$(*) \qquad r \leq x \text{ and } r \leq p_1 \vee \ldots \vee p_n \text{ imply } p_1, \ldots, p_n \leq x$$

*where $r, p_1, \ldots, p_n \in J$ and the inequality is minimal.*

*Proof:* The *'if'* part is proved as in lemma 3. Let $x$ be in $Cen(L)$. Then $x$ is standard and complemented. To prove $(*)$, notice that if $n = 1$, $r \leq p_1$ easily implies $p_1 \leq x$ (or otherwise $p_1 \leq \overline{x}$ — see the proof of lemma 8 — and $r \leq x \wedge \overline{x} = 0$). If $n > 1$, let $p_i \not\leq x$. Then $(x \vee p^i) \wedge (x \vee p_i) \wedge (p_i \vee p^i) \geq r$. By lemma 8 $x \wedge p_i = 0$; hence by neutrality $(x \wedge p^i) \vee (p_i \wedge p^i) \geq r$. If $x \wedge p^i = 0$, then $r \leq p^i$; hence $x \wedge p^i \neq 0$. Therefore, there exits $r' \in J$ such that $r' \leq x$ and $r' \leq p^i$. Notice that $p^i \not\leq x$ (otherwise we would have $p^i \geq r$). Thus, there exists a subset $P$ of $\{p_1, \ldots, p_n\} - \{p_i\}$ such that $r' \leq \vee P$, $r' \leq x$, the inequality is minimal and at least one $p_j \in P$ is not below $x$. Then we can repeat the process to further reduce the size of $P$. This process will eventually stop when the size of a new subset of $P$ is one, thus giving us $r'' \leq x$ such that $r'' \leq p_l$ and $p_l \not\leq x$, where $r''$ is a nonzero element of $J$. Since $p_l \leq \overline{x}$, $r'' \leq x \wedge \overline{x} = 0$, a contradiction. Lemma is proved. □

We are now in the position to prove the desired characterization of $Cen(L)$.

**Lemma 10** *$x \in Cen(L)$ iff the following hold:*
*(1) $x$ has a complement $\overline{x}$;*
*(2) for every $p \in J$, either $p \leq x$ or $p \leq \overline{x}$;*
*(3) if $p \in J$ and $p \leq a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m$ where $a_i \in J(x), b_j \in J(\overline{x})$, $i = 1, ..., n, j = 1, ..., m$ then either $p \leq a_1 \vee \ldots \vee a_n$ or $p \leq b_1 \vee \ldots \vee b_m$.*

*Remark*: It is easy to see that (2) and (3) guarantee the uniqueness of $\overline{x}$.
*Proof:* If (1), (2) and (3) hold, then the conditions of lemma 9 are proved similarly to the proof of lemma 3. Let $x \in Cen(L)$. Then (1) holds and we have already proved (2). If $p \leq x$, then sharpen the inequality in (3) by deleting elements until it becomes minimal and conclude $p \leq a_1 \vee \ldots \vee a_n$ by

applying $(*)$ of lemma 9. If $p \leq \overline{x}$, define $b$ as $b_1 \vee \ldots \vee b_m$. Then $(x \vee p) \wedge (x \vee b) \wedge (p \vee b) \geq p$. Since $p \leq \overline{x}$, $p \wedge x = 0$ and by neutrality of $x$: $(x \wedge b) \vee (p \wedge b) \geq p$. Therefore, $p \leq (x \wedge b) \vee (p \wedge b) \leq b$ which proves (3). □

Using the characterization given in lemma 10, one can rework the proof of lemma 5 to obtain

**Lemma 11** *$Cen(L)$ is a complete sublattice of $L$.* □

To finish the proof, we need

**Lemma 12** *$Cen(L)$ is atomistic.*

*Proof:* Let $c_p = \bigwedge(x \mid x \in Cen(L), x \geq p)$ where $p \in J$. By lemma 11, $c_p \in Cen(L)$. Every element of $Cen(L)$ is the join of such elements; in fact, $x = \bigvee(c_p \mid p \in J(x))$ for any $x \in Cen(L)$. Prove that $c_p$ is compact in $Cen(L)$. It is sufficient to prove that $c_p \leq \bigvee_{i \in I} c_{p_i}$ implies that $c_p$ is below the join of finitely many $c_{p_i}$'s. Since $p \leq c_p$, $p \leq \bigvee_{i \in I} c_{p_i}$. Hence, $p \leq \bigvee_{i \in I_f} c_{p_i}$ where $I_f$ is a finite subset of $I$ because $p$ is compact. $Cen(L)$ is a sublattice of $L$, therefore $\bigvee_{i \in I_f} c_{p_i} \in Cen(L)$ and this means $c_p \leq \bigvee_{i \in I_f} c_{p_i}$. Thus, $c_p$ is compact, and by the observation made earlier $Cen(L)$ is an algebraic lattice. Since $Cen(L)$ is Boolean, it is atomistic, in fact, of form $\mathbf{2}^X$ for some set $X$ [3]. □

# References


[1] M.K. Bennett, Separation conditions on convexity lattices, *Springer Lecture Notes in Mathematics*, **1149** (1984), 22–37.

[2] M.K. Bennett and G. Birkhoff, Convexity lattices, *Algebra Universalis* **20** (1985), 1–26.

[3] G. Birkhoff. *Lattice Theory*, 3rd ed., AMS, Providence, RI, 1967.

[4] N.D. Filippov, Projectivity of lattices, *Matem. Sb.* **70** (1966), 36–54.

[5] G. Grätzer. *General Lattice Theory.* Birkhäuser Verlag, Basel, 1978.

[6] L. Libkin, Direct product decompositions of lattices, closures and relation schemes, *Discrete Mathematics* **112** (1993), 119–138.

[7] L. Libkin, I. Muchnik, The lattice of subsemilattices of a semilattice, *Algebra Universalis*, **31** (1994), 252–255.

[8] G. Richter, On the structure of lattices in which every element is a join of join-irreducible elements, *Periodica Mathematica Hungarica* **13** (1982), 47–69.