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An elementary proof that upper and lower powerdomain constructions commute

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1 Introduction

It was proved in [1] that lower and upper powerdomain constructions commute on all domains. In that proof, domains were represented as information systems. In [2] a rather complicated algebraic proof was given which relied on universality properties of powerdomains proved in the previous works of the author of [2]. Here we give an elementary algebraic proof that upper and lower powerdomain constructions commute. The proof is essentially a reduction of the problem to establishing a 1-1 correspondence between certain disjunctive and conjunctive normal forms.

2 Definitions

A subset X of a partially ordered set is called *directed* if a common upper bound exists for any two elements of X, i.e. given $x_1, x_2 \in X$, there exists $x \in X$ such that $x \geq x_1, x_2$. A poset is called complete (abbreviated – *cpo*) if every directed subset has a least upper bound. An element of a cpo is called *compact* if it can not be below a least upper bound of a directed set X without being below an element of X. A cpo is called algebraic if every element is the least upper bound of compact elements below it, see [3].

A *domain* in this paper is an algebraic cpo with bottom. Given a domain D, $\leq$ denotes its order and $\mathbf{K}D$ is the set of its compact elements. Given $A, B \subseteq D$, *lower* and *upper powerdomain orderings* are given by

$$A \sqsubseteq^{\flat} B \Leftrightarrow \forall a \in A \exists b \in B : \ a \leq b$$

$$A \sqsubseteq^{\sharp} B \Leftrightarrow \forall b \in B \exists a \in A : \ a \leq b$$

A subset of an ordered set is called an antichain if no two elements in it are comparable. If $\langle X, \leq \rangle$ is an ordered set and $Y \subseteq X$, then $\max_{\leq} Y$ and $\min_{\leq} Y$ are sets of maximal and minimal elements of Y. We will use just $\max Y$ and $\min Y$ if the ordering is understood. $A_{fin}(X)$ stands for the

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set of all finite antichains of X. The lower and upper powerdomains are defined to be the ideal completions of $\langle A_{fin}(\mathbf{K}D), \sqsubseteq^\flat\rangle$ and $\langle A_{fin}(\mathbf{K}D), \sqsubseteq^\sharp\rangle$ respectively. They are denoted by $\mathcal{P}^\flat(D)$ and $\mathcal{P}^\sharp(D)$. $\langle A_{fin}(\mathbf{K}D), \sqsubseteq^\flat\rangle$ and $\langle A_{fin}(\mathbf{K}D), \sqsubseteq^\sharp\rangle$ are posets of compact elements of $\mathcal{P}^\flat(D)$ and $\mathcal{P}^\sharp(D)$ [3].

Remark: A traditional definition of the powerdomain construction is the ideal completion of $P_{fin}(\mathbf{K}D)$, the set of all finite subsets of $\mathbf{K}D$. The two can be easily shown to be equivalent. We prefer to work with antichains because $\sqsubseteq^\flat$ and $\sqsubseteq^\sharp$ are partial orders on $A_{fin}(\mathbf{K}D)$ but only preorders on $P_{fin}(\mathbf{K}D)$.

Our goal is to prove

Theorem *For any domain D, $\mathcal{P}^\sharp(\mathcal{P}^\flat(D))$ and $\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ are isomorphic.*

The proof is constructive, i.e. an isomorphism and its inverse are explicitly described.

3 Proof

To prove that two domains D_1 and D_2 are isomorphic, i.e. that there exists a pair of continuos mutually inverse maps between D_1 and D_2, it is enough to prove that $\mathbf{K}D_1$ and $\mathbf{K}D_2$ are isomorphic as posets, i.e. that there exists a pair of monotone mutually inverse maps between $\mathbf{K}D_1$ and $\mathbf{K}D_2$.

A compact element of $\mathcal{P}^\sharp(\mathcal{P}^\flat(D))$ is a finite antichain, w.r.t. $\sqsubseteq^\flat$, of finite antichains of compact elements of D, and a compact element of $\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ is a finite antichain, w.r.t. $\sqsubseteq^\sharp$, of finite antichains of compact elements of D. Given a finite set of finite sets $\mathcal{A} = \{A_1, \ldots, A_n\}$ where $A_i = \{a^i_1, \ldots, a^i_{k_i}\}$, let $F_\mathcal{A}$ be the set of functions $f : \{1, ..., n\} \to \mathbb{N}$ such that for any i: $1 \leq f(i) \leq k_i$. For $f \in F_\mathcal{A}$, let $f(\mathcal{A}) = \{a^i_{f(i)} \mid i = 1, \ldots, n\}$. If all A_i's are subsets of D, define two maps ϕ and ψ as follows:

$$\phi(\mathcal{A}) = \min_{f\in F_\mathcal{A}} {}_{\sqsubseteq^\flat}(\max f(\mathcal{A}))$$

$$\psi(\mathcal{A}) = \max_{f\in F_\mathcal{A}} {}_{\sqsubseteq^\sharp}(\min f(\mathcal{A}))$$

Now, we claim that ψ maps $\mathbf{K}\mathcal{P}^\sharp(\mathcal{P}^\flat(D))$ to $\mathbf{K}\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ and ϕ maps $\mathbf{K}\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ to $\mathbf{K}\mathcal{P}^\sharp(\mathcal{P}^\flat(D))$ and, moreover, these maps establish the desired isomorphism, i.e. they are mutually inverse and monotone. The first claim follows immediately from the definitions of ϕ and ψ. To complete the proof, it is enough to show that ϕ is monotone and $\phi \circ \psi = \mathsf{id}$. The proof of monotonicity of ψ and $\psi \circ \phi = \mathsf{id}$ is dual. We start with two easy observations:

Lemma *Let Y_1, Y_2 be finite subsets of an arbitrary poset X. Then*
1) $Y_1 \sqsubseteq^\flat Y_2$ iff $\max Y_1 \sqsubseteq^\flat \max Y_2$;
2) $Y_1 \sqsubseteq^\sharp Y_2$ iff $\min Y_1 \sqsubseteq^\sharp \min Y_2$. □

Claim 1: ϕ is monotone.
Proof of claim 1: Let $\mathcal{A}, \mathcal{B} = \{B_1, ..., B_m\} \in \mathbf{K}\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ and $\mathcal{A} \sqsubseteq^\flat \mathcal{B}$. We must prove $\phi(\mathcal{A}) \sqsubseteq^\sharp \phi(\mathcal{B})$. In view of lemma, it is enough to show that for any $f \in F_\mathcal{B}$ there exists $g \in F_\mathcal{A}$ such that $g(\mathcal{A}) \sqsubseteq^\flat f(\mathcal{B})$. Since for each $i = 1, \ldots, n$ there exists j_i such that $A_i \sqsubseteq^\sharp B_{j_i}$, there is an element $a^i_{p_i} \in A_i$ such that $a^i_{p_i} \leq b^{j_i}_{f(j_i)}$. Let $g(i) = p_i$. Then for this function g one has $\{a^i_{g(i)} \mid i = 1, \ldots, n\} \sqsubseteq^\flat \{b^i_{f(i)} \mid i = 1, \ldots, m\}$, i.e. $g(\mathcal{A}) \sqsubseteq^\flat f(\mathcal{B})$. Claim 1 is proved.

Let $\mathcal{A} \in \mathbf{K}\mathcal{P}^\flat(\mathcal{P}^\sharp(D))$ and $\mathcal{B} = \{B_1, \ldots, B_m\} = \phi(\mathcal{A}) \in \mathbf{K}\mathcal{P}^\sharp(\mathcal{P}^\flat(D))$. In view of lemma, to show that $\psi \circ \phi = \mathsf{id}$, i.e. that $\psi(\mathcal{B}) = \mathcal{A}$, it suffices to prove
Claim 2: For any $f \in F_\mathcal{B}$ there exists $A_i \in \mathcal{A}$ such that $f(\mathcal{B}) \sqsubseteq^\sharp A_i$.
Claim 3: Every A_i is in $\psi(\mathcal{B})$.

Proof of claim 2: Let $\mathcal{C}$ be the collection of all sets $f(\mathcal{A})$ where $f \in F_\mathcal{A}$; $\mathcal{C} = \{C_1, \ldots, C_k\}$. Then for any $g \in F_\mathcal{C}$, there exists $A_i \in \mathcal{A}$ such that A_i is contained in $g(\mathcal{C})$ because, if this is not the case, for any $A_i \in \mathcal{A}$ there exists $j_i \leq k_i$ such that $a^i_{j_i} \in A_i$ and, for any $f \in F_\mathcal{A}$, g on $f(\mathcal{A})$ picks an element different from $a^i_{j_i}$. If we define f_0 such that $f_0(i) = j_i$, g may pick only elements of form $a^i_{j_i}$ on $f_0(\mathcal{A})$, a contradiction. Therefore, $g(\mathcal{C}) \sqsubseteq^\sharp A_i$ for some i.

Let $f \in F_\mathcal{B}$. Let H be the set of functions in $F_\mathcal{A}$ that correspond to elements of $\mathcal{B} = \phi(\mathcal{A})$ or, in other words, $\max h(\mathcal{A}) \in \mathcal{B}$ for $h \in H$. Then, for any $h' \in F_\mathcal{A} - H$, there exists a function $h \in H$ such that $\max h(\mathcal{A}) \sqsubseteq^\flat \max h'(\mathcal{A})$, i.e. $h(\mathcal{A}) \sqsubseteq^\flat h'(\mathcal{A})$. Since $h \in H$, $\max h(\mathcal{A}) \in \mathcal{B}$, i.e. $\max h(\mathcal{A}) = B_i$. If $f(i) = j$, then there is an element in $h'(\mathcal{A})$ that is greater than b^i_j. Define a function $g \in F_\mathcal{C}$ to coincide with f on those C_i's that are given by functions in H. On C_i that corresponds to $f \in F_\mathcal{A} - H$, let g pick an element which is greater than some b^i_j where $f(i) = j$ (we have just shown it can be done). Then $f(\mathcal{B}) \sqsubseteq^\sharp \{c^i_{g(i)} \mid i = 1, \ldots, k\} = g(\mathcal{C})$. We know that there exists $A_i \in \mathcal{A}$ such that $g(\mathcal{C}) \sqsubseteq^\sharp A_i$. Thus, $f(\mathcal{B}) \sqsubseteq^\sharp A_i$. Claim 2 is proved.

Proof of claim 3: Prove that for any $a^i_j \in A_i$ there exists $B_l \in \mathcal{B}$ such that $a^i_j \in B_l$. Consider the set $F^{ij}_\mathcal{A}$ of functions $f \in F_\mathcal{A}$ such that $f(i) = j$. If for no $f \in F^{ij}_\mathcal{A}$: $a^i_j \in \max f(\mathcal{A})$, then there exists $A_p \in \mathcal{A}$ such that all elements of A_p are greater than a^i_j, i.e. $A_i \sqsubseteq^\sharp A_p$ which contradicts our assumption that $\mathcal{A}$ is an antichain w.r.t. $\sqsubseteq^\sharp$. Hence, $a^i_j \in \max f(\mathcal{A})$ for at least one function in $F^{ij}_\mathcal{A}$. Since $\mathcal{A}$ is an antichain, for any $p \neq i$ there exists $a^p_q \in A_p$ which is not greater than any element of A_i. Change f to pick such an element for any $p \neq i$. Then a^i_j is still in $\max f(\mathcal{A})$. There exists a function $f' \in F_\mathcal{A}$ such that $\max f'(\mathcal{A}) \sqsubseteq^\flat \max f(\mathcal{A})$ and $\max f'(\mathcal{A}) \in \phi(\mathcal{A})$. If $f'(i) = j' \neq j$, then, since $f'(\mathcal{A}) \sqsubseteq^\flat f(\mathcal{A})$ and A_i is an antichain, $a^i_{j'} \leq a^p_q$ for some p and q, where $p \neq i$. But this contradicts the definition of f. Hence, $f'(i) = j$ and $a^i_j \in \max f'(\mathcal{A})$ because $a^i_j \in \max f(\mathcal{A})$. Since $\max f'(\mathcal{A}) = B_l$ for some index l, $a^i_j \in B_l \in \mathcal{B}$.

Let $\mathcal{B}'$ be the collection of elements of $\mathcal{B}$ that contain elements of A_i. Then we can define a function $f \in F_\mathcal{B}$ on elements of $\mathcal{B}'$ to pick all elements of A_i. Each $B_j \in \mathcal{B} - \mathcal{B}'$ either contains an element of A_i or contains an element which is greater than some $a^i_p \in A_i$. Let f pick any such element. Then $\min f(\mathcal{B}) = A_i$. Suppose $A_i \notin \psi(\mathcal{B})$. Then $A_i \sqsubseteq^\sharp \min g(\mathcal{B})$ for some function $g \in F_\mathcal{B}$ such that $\min g(\mathcal{B}) \in \psi(\mathcal{B})$. By claim 2, $g(\mathcal{B}) \sqsubseteq^\sharp A_j$ for some A_j. Hence, $\min g(\mathcal{B}) \sqsubseteq^\sharp A_j$ and since $\mathcal{A}$ is an antichain w.r.t. $\sqsubseteq^\sharp$, $A_i = A_j = \min g(\mathcal{B}) \in \psi(\mathcal{B})$. This finishes the proof of claim 3 and the theorem.

References

[1] K.E. Flannery, J.J. Martin, Hoare and Smyth power domain constructors commute under composition, *J. Comput. Syst. Sci.* **40** (1990), 125-135.

[2] R. Heckmann, Lower and upper power domain constructions commute on all cpos, *Inform. Process. Letters* **40** (1991), 7-11.

[3] C.A. Gunter and D.S. Scott. Semantic domains, In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 633–674. North Holland, 1990.