LOGIC & AUTOMATA — ASSIGNMENT 1

1. (*7 marks*) Consider MSO formulae over a vocabulary that has only one binary relation symbol $<$. We are interested in structures where $<$ is interpreted as a linear order, i.e. we are looking at structures $\langle A, < \rangle$, where A is a set, and $<$ is a linear order on it.

   Prove that there is no MSO sentence that checks whether the cardinality of A is a perfect square, i.e. of the form n^2 for some $n \in \mathbb{N}$.

   Hint: Use Büchi's theorem for words.

2. (*8 marks*) Consider the following nondeterministic automaton model on words: $(Q, q_0, F, \delta_\rightarrow, \delta_\leftarrow)$, where Q is a set of states, q_0 is a single initial state, states in $F \subseteq Q$ are final, and $\delta_\rightarrow, \delta_\leftarrow : Q \times \Sigma \to 2^Q$ are two transition functions.

   The automaton starts in state q_0, reading the first symbol of a word. Every time the automaton is in a state q, reading symbol a_i of a word $a_0 \dots a_{n-1}$, it nondeterministically chooses one of the two transition functions and does the following:

   - For $\delta_\rightarrow$: it selects a state $q' \in \delta_\rightarrow(q, a)$, and moves one position to the right, i.e. it now reads a_{i+1} in state q'. In the special case when $i = n - 1$ (i.e. there are no more symbols on the right), the automaton accepts if $q' \in F$ and rejects otherwise.
   - For $\delta_\leftarrow$: it selects a state $q' \in \delta_\leftarrow(q, a)$, and moves one position to the left, i.e. it now reads a_{i-1} in state q'. In the special case when $i = 0$ (i.e. there are no more symbols on the left), the automaton rejects.

   Thus, these automata can go back-and-forth, like Turing machines. But we further impose a condition that such an automaton can visit a position *at most twice* (if it visits any position for a third time, it rejects).

   These automata clearly generalize the usual NFAs (when one transition function is empty), so they accept all regular languages. Your goal is to prove the converse *using Büchi's theorem*: every language they accept is regular.

3. (*15 marks*) Some notation we shall need: If $T = (D, \lambda)$ is a binary tree, the *frontier word* of this tree, $fr(T)$, is the sequence of labels of its leaves, in the lexicographic order. More precisely, if the leaves of T are $s_1, \dots, s_n$ such that $s_1 < \dots < s_n$, where $<$ is the usual lexicographic order on $\{0, 1\}^*$, then

   $$fr(T) \;=\; \lambda(s_1) \dots \lambda(s_n).$$

   If L is a tree language (i.e. a set of Σ-labeled binary trees), we write $Fr(L)$ for the word language $\{fr(T) \mid T \in L\} \subseteq \Sigma^*$.

   Each of the following is worth 5 marks.

   (a) Let L' be a word language. Use Büchi's and Thatcher-Wright's theorems to prove that there is a regular tree language L such that $L' = Fr(L)$.

   (b) Given an example of a regular tree language L such that $Fr(L)$ is not a regular word language (and explain why it is not).

   (c) Consider a restriction of MSO over binary trees in which all quantifiers $\exists X$ range only over *antichains*: i.e. sets X of nodes such that no $s \in X$ is a prefix of another node $s' \in X$.

   Prove that every regular tree language (of binary trees) can be defined in this restriction of MSO.