# The local structure of semi-sparse permutations

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# Permutations

Permutation of length $n$: an ordering of $1, \ldots, n$.

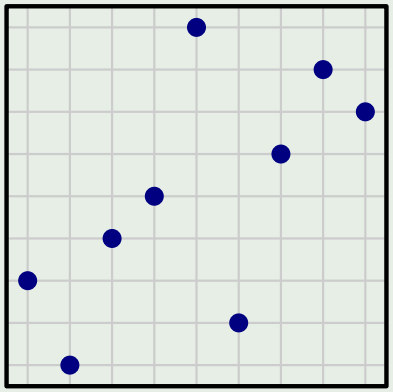
## Example

$$\pi = 314592687 \in S_9$$

The plot of a permutation: $\{(i, \pi(i)) : 1 \leqslant i \leqslant n\} \subset \mathbb{R}^2$.

- One point in each row; one point in each column.

## Example (314592687)



# Inversions

Inversion: A pair of NW–SE points.

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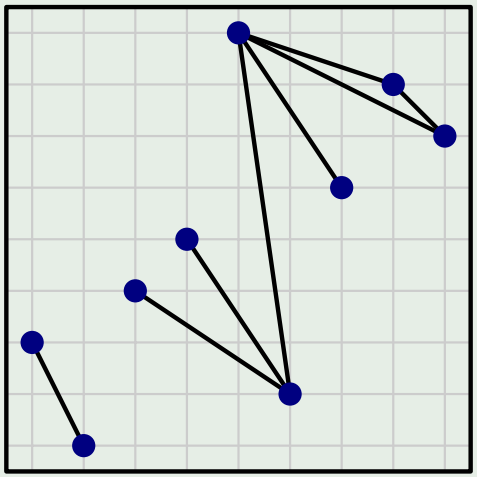
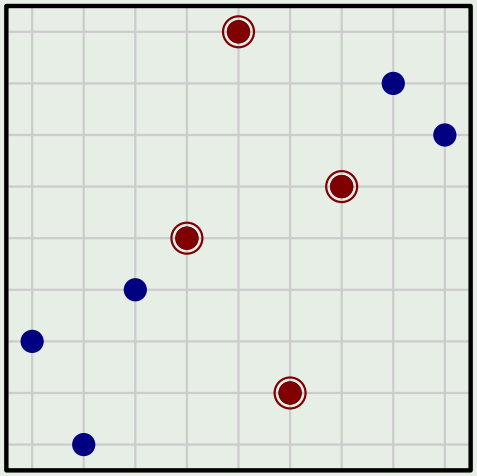


314592687 has 8 inversions.

# Patterns

## Example (314592687)



The consecutive pattern 2413 occurs at position 4.

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- What does a *typical* large $n$-permutation with $m(n)$ inversions look like, for some function $m(n)$?

$$m(n) = \rho n^2 \qquad \text{— dense}$$

$$m(n) = n^{3/2} \qquad \text{— semi-sparse}$$

$$m(n) = \alpha n \qquad \text{— sparse}$$

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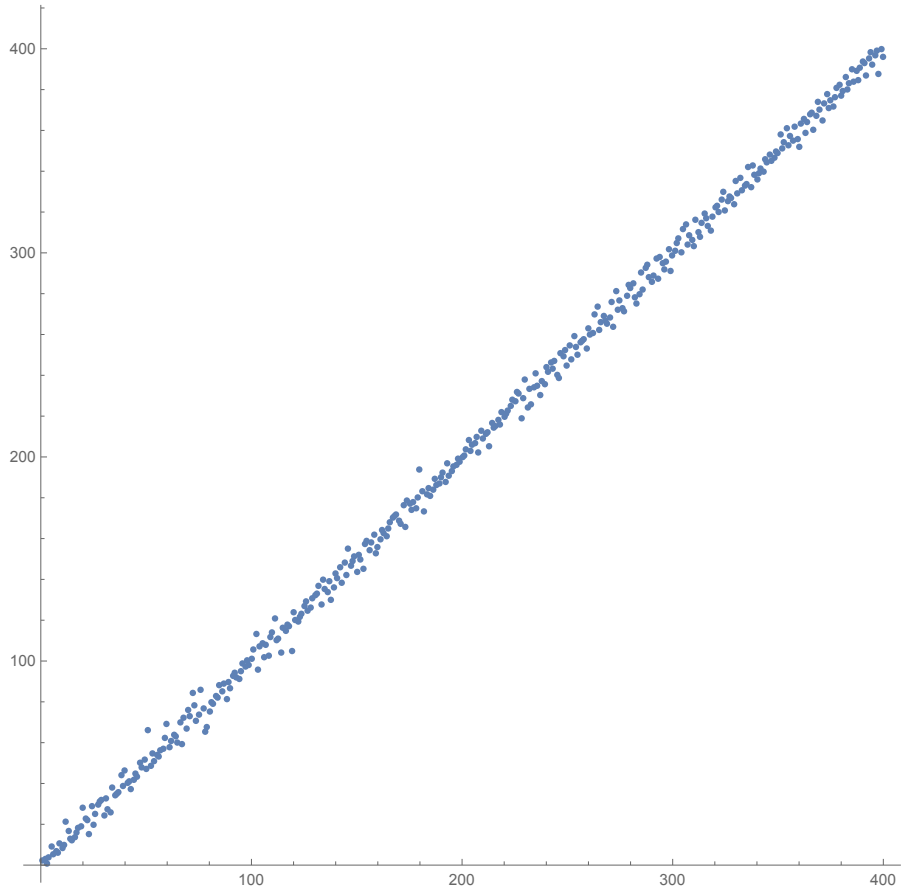
- What properties of an $n$-permutation with $m(n)$ inversions hold *asymptotically almost surely*?

## Definition

A property $Q$ holds *asymptotically almost surely* (*a.a.s.*) if $\lim_{n\to\infty} \mathbb{P}[Q] = 1$.

# Evolution




$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.01$

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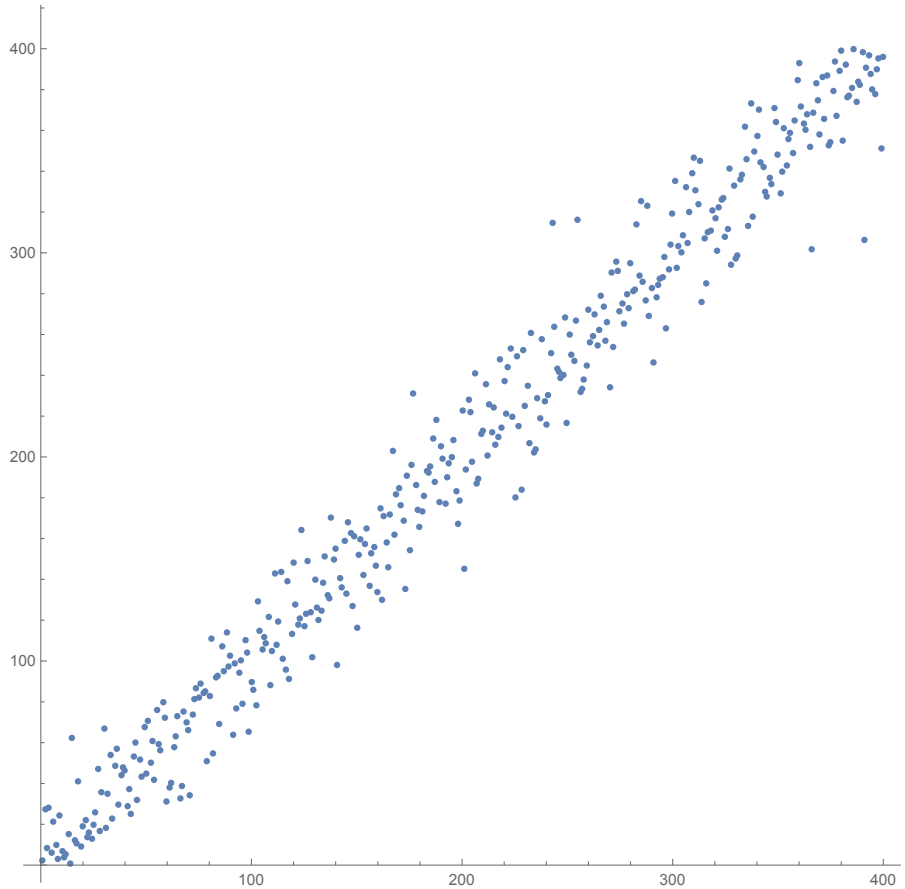


$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.05$

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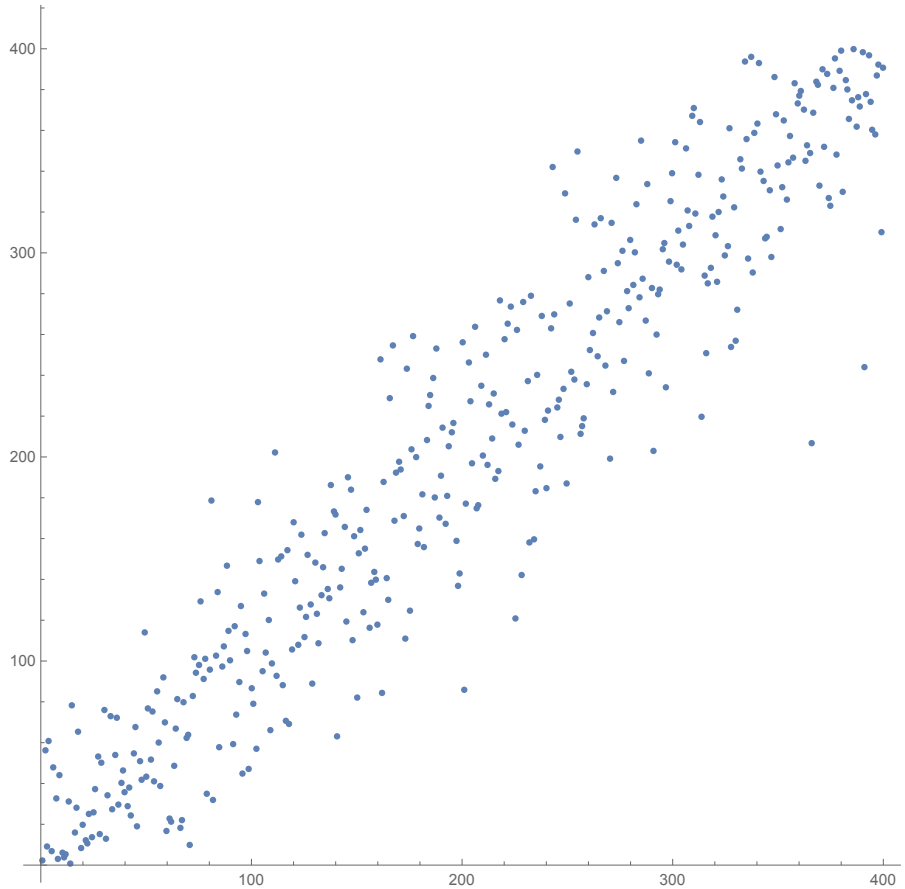
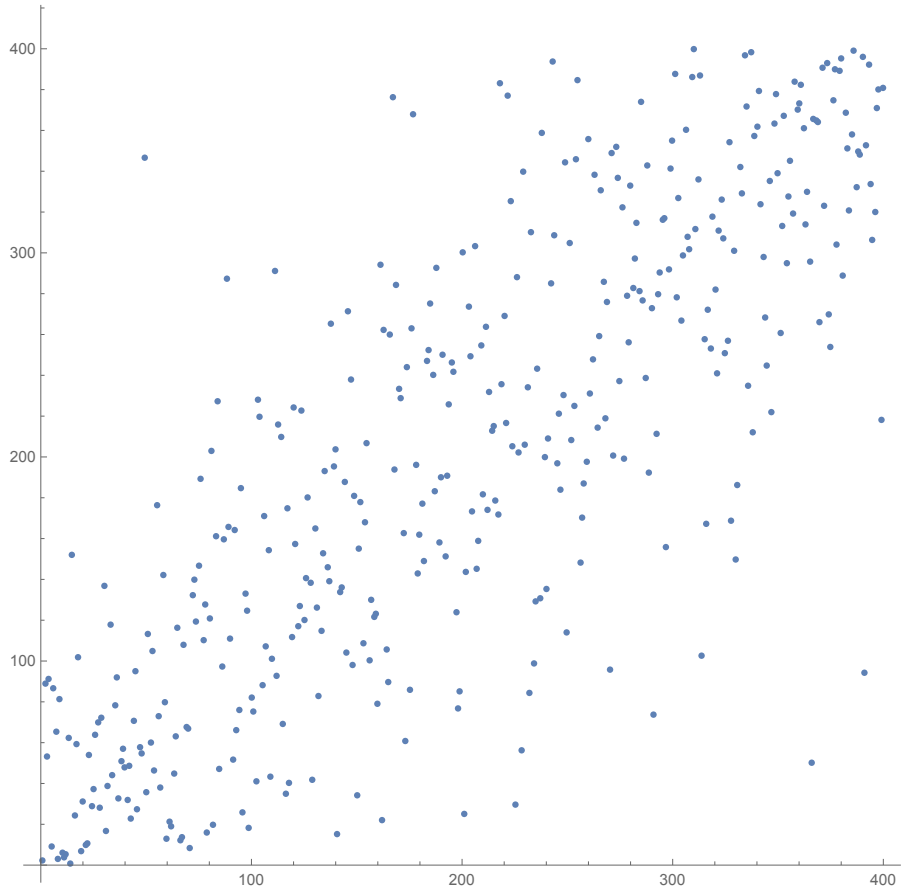


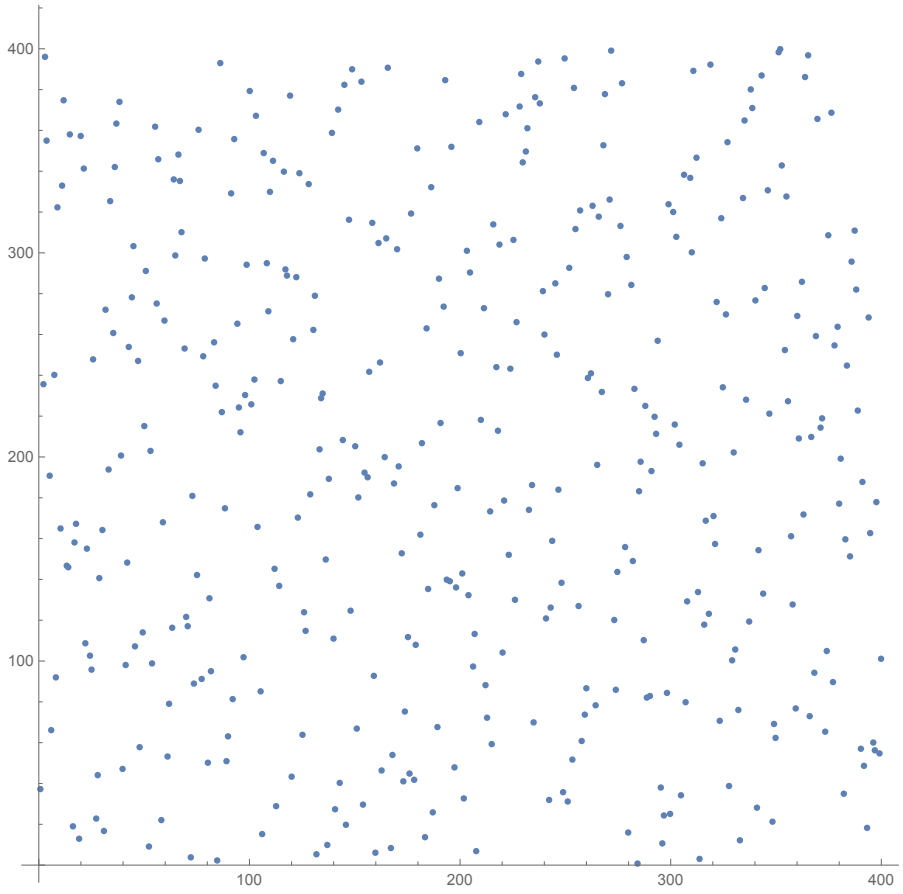

$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.10$

# Evolution




$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.20$

# Evolution




$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.50$

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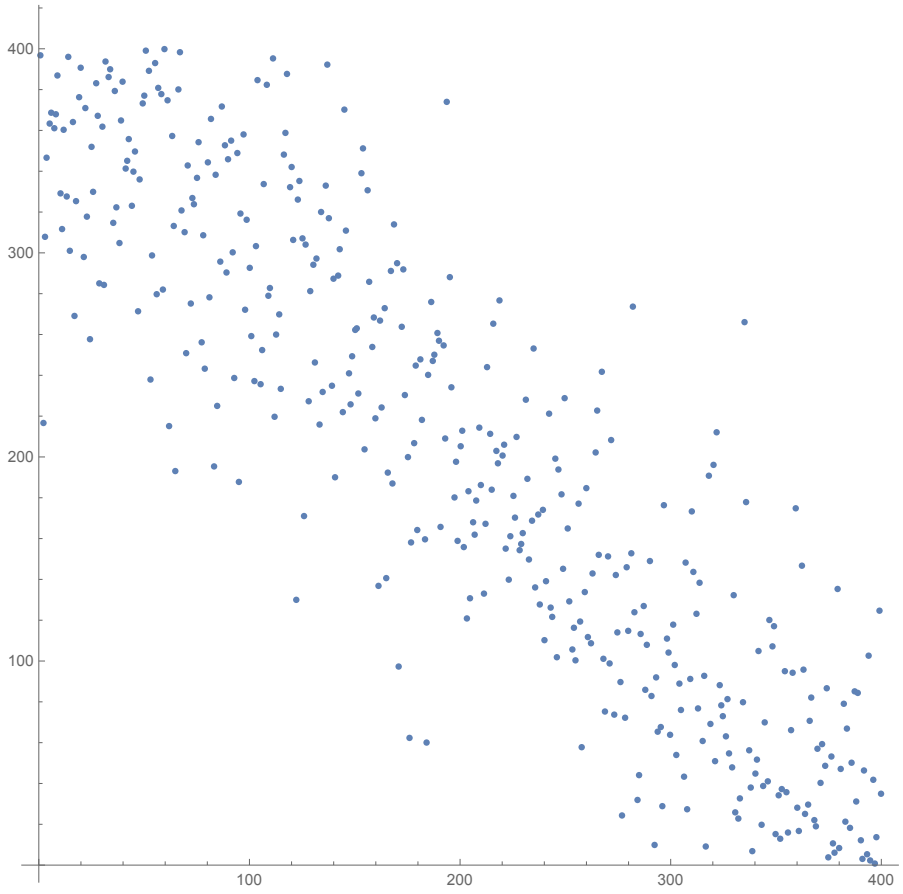
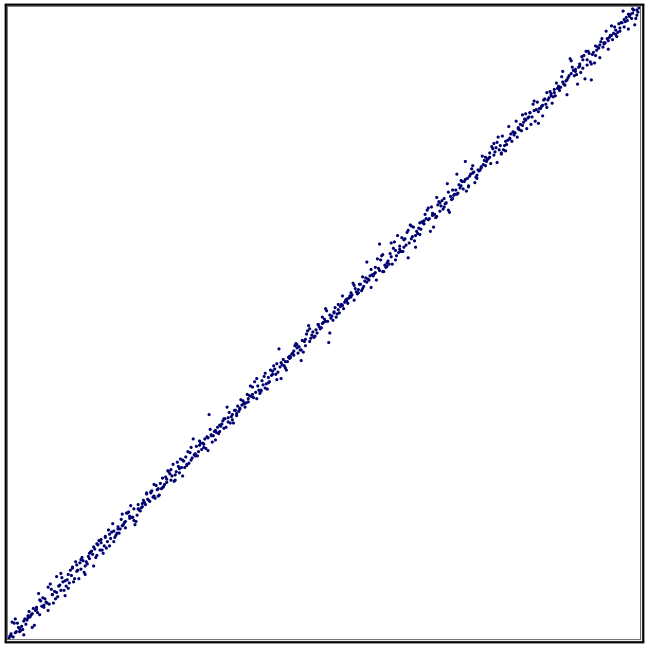



$n = 400$ $\qquad$ $m/\binom{n}{2} = 0.85$

# Semi-sparse permutations

A semi-sparse permutation has "few" inversions: $n \ll m \ll n^2$.



Almost all the points are close to the main diagonal.

**Definition ("$y$ grows faster than $x$")**

We write $x \ll y$ or $y \gg x$ if $\lim_{n\to\infty} x/y = 0$.

# Specific questions: local structure

A semi-sparse permutation has $n \ll m \ll n^2$.

$$S_{n,m} \;=\; \{\sigma \in S_n : \mathsf{inv}(\sigma) = m\}$$

Select $\boldsymbol{\sigma}_n$ uniformly from $S_{n,m}$ and $i < j$ uniformly from $[n]$. Then,

$$\mathbb{P}\big[\boldsymbol{\sigma}_n(i) > \boldsymbol{\sigma}_n(j)\big] \;=\; m/\tbinom{n}{2} \;\to\; 0 \text{ as } n \to \infty.$$

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## Questions

What does a semi-sparse permutation look like when we ***zoom in***?

- $\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n(j) > \boldsymbol{\sigma}_n(j+1)\big]$? — *expected number of* descents

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- $\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n(j) > \boldsymbol{\sigma}_n(j+d)\big]$, for a given $d$? — $d$-descents
- $\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_n(j+1)\ldots\boldsymbol{\sigma}_n(j+d) \text{ is } \pi\big]$, for given $\pi \in S_d$? — patterns

# Results

## Theorem

Suppose $n \ll m \ll n^2/\log^2 n$. Select $\boldsymbol{\sigma}_n$ uniformly from $S_{n,m}$.

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2. If $d = d(n) \ll \sqrt{m/n}$ and $\pi_n \in S_d$, then

$$\mathbb{P}\big[\boldsymbol{\sigma}_n(j+1)\ldots\boldsymbol{\sigma}_n(j+d) \text{ is } \pi_n\big] \;\sim\; 1/d!.$$

# Results

## Theorem (local uniformity)

Suppose $n \ll m \ll n^2/\log^2 n$. Select $\boldsymbol{\sigma}_n$ uniformly from $S_{n,m}$.

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## Local-global dichotomy

- Expected number of descents is asymptotically $(n-1)/2$.
- Permutations from $S_{n,m}$ are *locally uniform*.
- Local structure reveals nothing about global structure.

# Local structure: consecutive patterns

$$S_{n,m} \;=\; \{\sigma \in S_n : \mathsf{inv}(\sigma) = m\}$$

**Proposition (position-independence)**

*Within $S_{n,m}$, for any consecutive pattern $\pi$ and positive $i, j \leqslant n + 1 - |\pi|$,*

$$\mathbb{P}\big[\pi \textit{ occurs at position } i\big] \;=\; \mathbb{P}\big[\pi \textit{ occurs at position } j\big].$$

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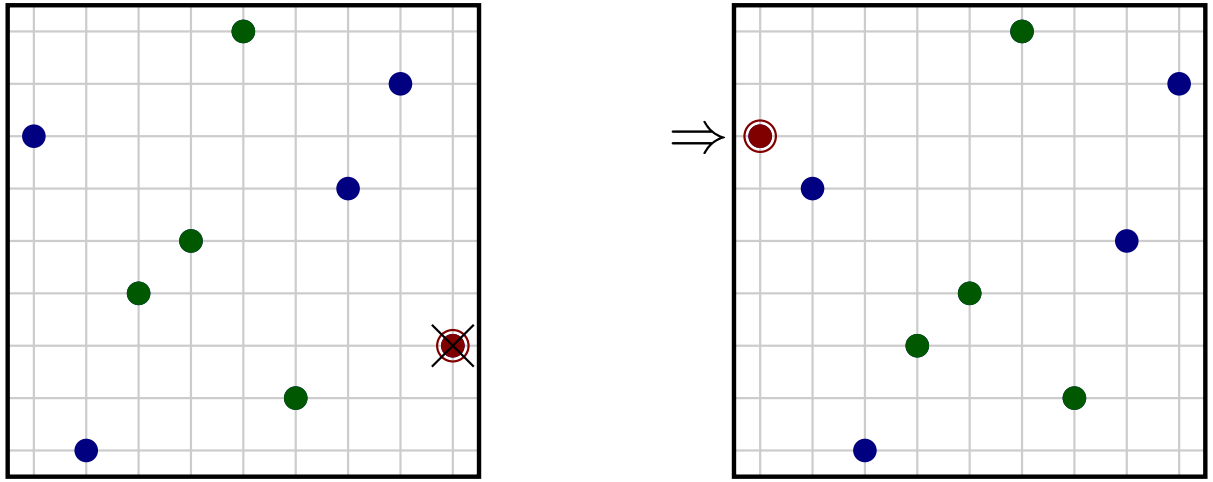
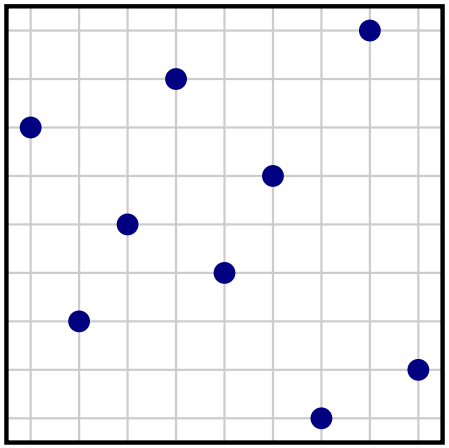
*A pictorial proof*:



This is a bijection on $S_{n,m}$ that shifts consecutive patterns right by one.

# Inversion sequences

Inversion sequence of $\sigma$: $(e_j)$, where $e_j = \left|\{i : i < j \text{ and } \sigma(i) > \sigma(j)\}\right|$.



0 1 1 0 3 2 6 0 7

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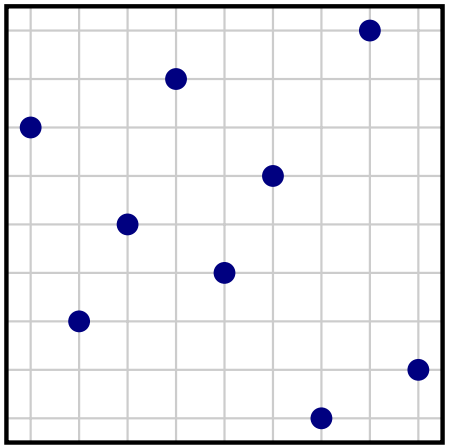

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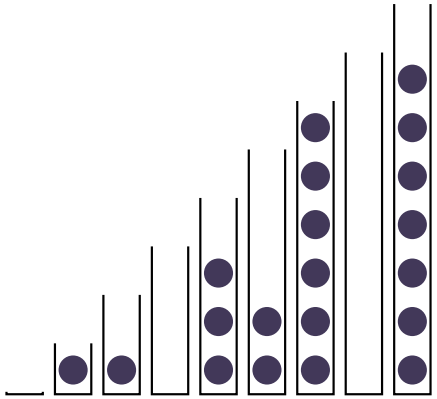
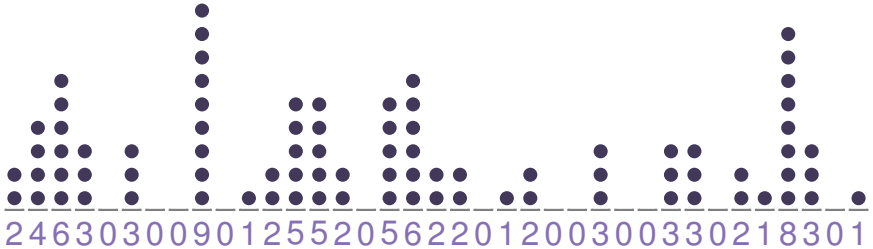


Balls-in-boxes: Nonnegative sequences $(e_j)_{j=1}^n$ with $e_j < j$ whose sum is $m$ are in bijection with $n$-permutations that have $m$ inversions.

# Unrestricted balls-in-boxes

Weak compositions of $m$ with $n$ parts, $\mathcal{C}_{n,m}$:



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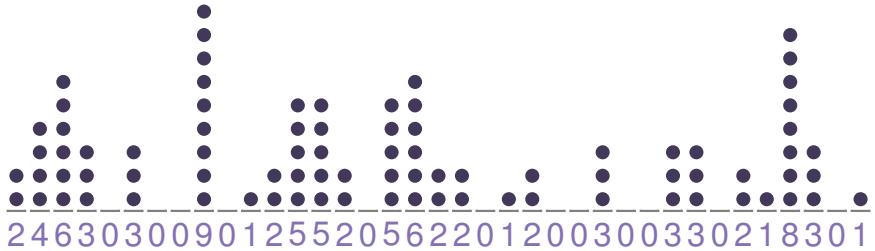


## Proposition

*If* $n \ll m \ll n^2/\log n$, *then for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\textit{every box has at most } \; r_{n,m} = \tfrac{m}{n}(1+\varepsilon)\log n \; \textit{ balls}\right] = 1.$$

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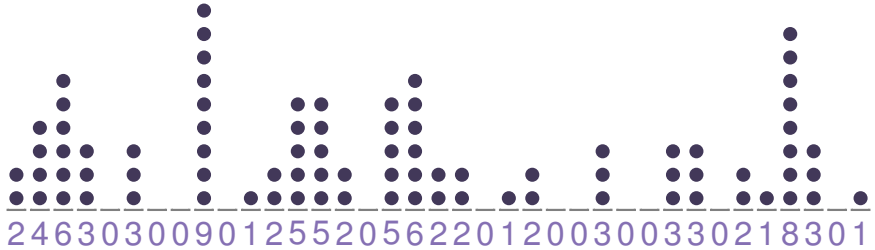


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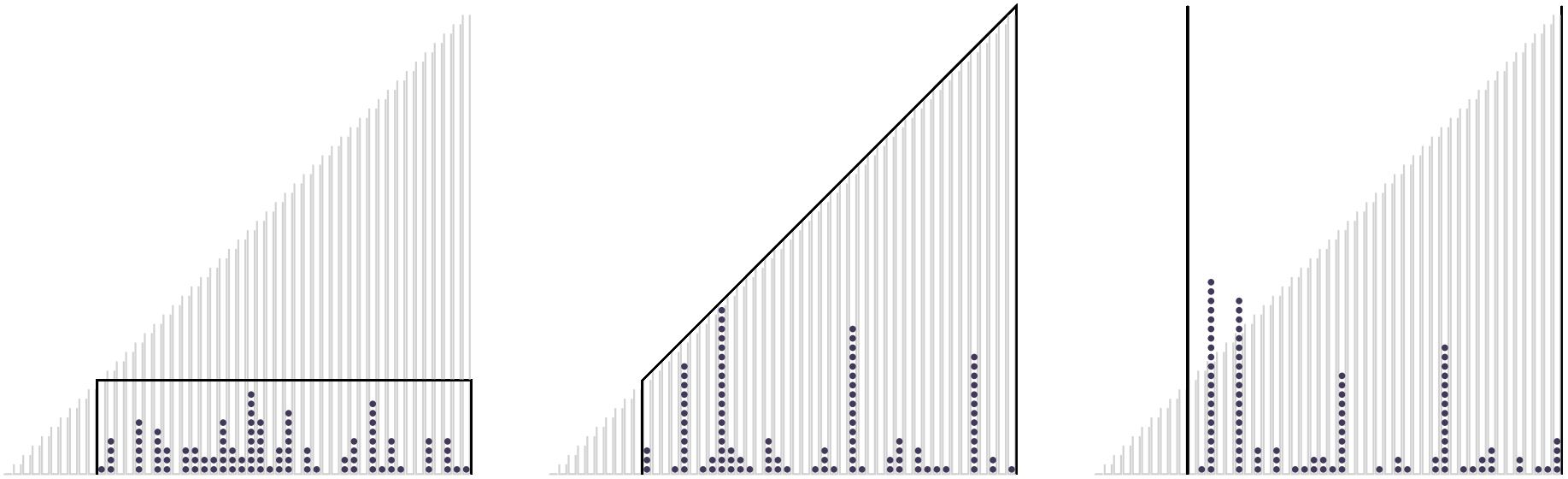
## Corollary

If $\mathcal{C}^{\mathsf{R}}_{n,m}$ is the set of restricted compositions with at most $r_{n,m}$ balls per box, then $|\mathcal{C}^{\mathsf{R}}_{n,m}| \sim |\mathcal{C}_{n,m}|$ (that is, $\lim_{n\to\infty} |\mathcal{C}^{\mathsf{R}}_{n,m}|/|\mathcal{C}_{n,m}| = 1$).

# Approximating inversion sequences



$\mathcal{C}^{\mathsf{R}}_{n,m}$ $\mathcal{C}^{\mathsf{S}}_{n,m}$ $\mathcal{C}_{n,m}$

Since,

$$\mathcal{C}^{\mathsf{R}}_{n,m} \subset \mathcal{C}^{\mathsf{S}}_{n,m} \subset \mathcal{C}_{n,m}$$

and

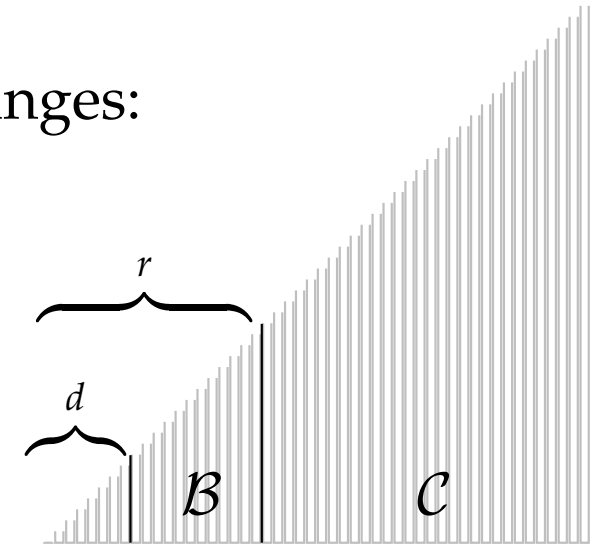
$$|\mathcal{C}^{\mathsf{R}}_{n,m}| \sim |\mathcal{C}_{n,m}|,$$

we also have

$$|\mathcal{C}^{\mathsf{S}}_{n,m}| \sim |\mathcal{C}_{n,m}| = \binom{m+n-1}{m}.$$

# Counting

Split boxes into three ranges:




Let $B = \binom{r}{2} - \binom{d}{2}$ be the capacity of $\mathcal{B}$ (boxes $d+1, \ldots, r$).

Let $b_k$ be the number of ways of placing $k$ balls in $\mathcal{B}$.

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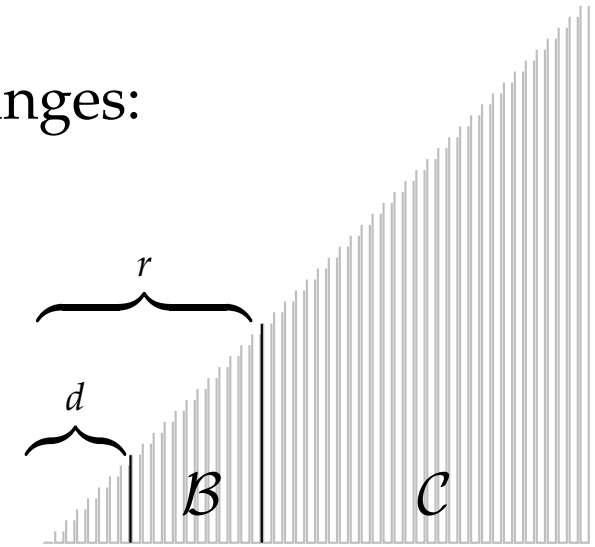


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Number of permutations with the first $d$ boxes empty (increasing):

$$\sum_{k=0}^{B} b_k \left|\mathcal{C}^{\mathsf{S}}_{n-r,m-k}\right| \sim \sum_{k=0}^{B} b_k \left|\mathcal{C}_{n-r,m-k}\right|$$

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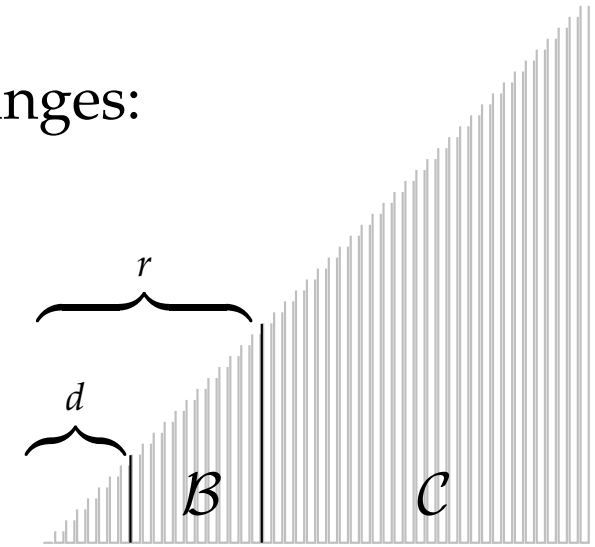


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Number of permutations with the first $d$ boxes full (decreasing):

$$\sum_{k=0}^{B} b_k \left|\mathcal{C}^{\mathsf{S}}_{n-r,m-\binom{d}{2}-k}\right| \sim \sum_{k=0}^{B} b_k \left|\mathcal{C}_{n-r,m-\binom{d}{2}-k}\right|$$

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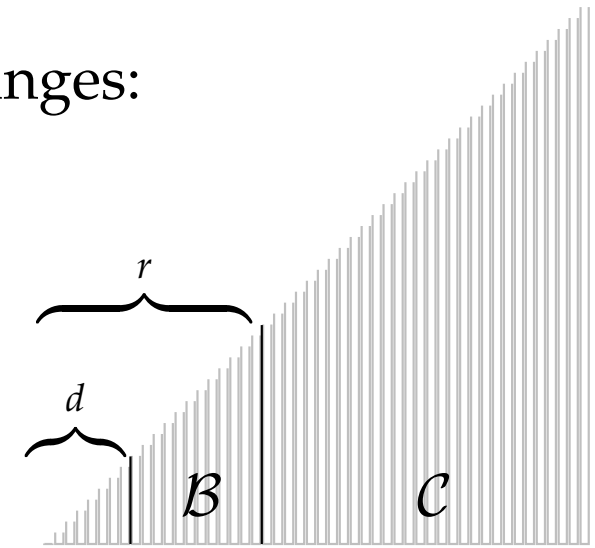


If $\binom{d}{2} \ll m/n$, then

$$\mathbb{P}\big[\boldsymbol{\sigma}_n[1,d] \text{ is increasing}\big] \sim \mathbb{P}\big[\boldsymbol{\sigma}_n[1,d] \text{ is } \pi_n\big] \sim \mathbb{P}\big[\boldsymbol{\sigma}_n[1,d] \text{ is decreasing}\big]$$

So by position independence,

$$\mathbb{P}\big[\boldsymbol{\sigma}_n(j+1)\ldots\boldsymbol{\sigma}_n(j+d) \text{ is } \pi_n\big] \sim \frac{1}{d!}.$$

# Further results

## Theorem (critical window)

Suppose $n \ll m \ll n^2/\log^2 n$. Select $\boldsymbol{\sigma}_n$ uniformly from $S_{n,m}$.

If $d \sim \alpha\sqrt{m/n}$ and $\pi_n \in S_d$ with $\mathsf{inv}(\pi_n) \sim \rho d^2/2$, then

$$\mathbb{P}\big[\boldsymbol{\sigma}_n(j+1)\ldots\boldsymbol{\sigma}_n(j+d) \text{ is } \pi_n\big] \;\sim\; e^{(1-2\rho)\alpha^2/4}\frac{1}{d!}.$$

# Further questions

What about the rest of the semi-sparse range? $n^2/\log^2 n \ll m \ll n^2$
— How can we approximate?

What about dense permutations? $m \sim \alpha n^2$
— Do *permutons* help?

# Thanks for listening!