Resilience of random graphs with respect to Hamiltonicity.

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joint work with
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April 2019
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**Definition (Local resilience)**

The local resilience of a graph $G$ with respect to some property $\mathcal{P}$ is the maximum number $r$ such that for any subgraph $H \subseteq G$ with $\Delta(H) < r$, the graph $G \setminus H$ satisfies $\mathcal{P}$.

**This talk:** $G$ will be random and $\mathcal{P}$ will be Hamiltonicity.
Dirac’s theorem: resilience version

Theorem (Dirac, 1952)

If $G$ is an $n$-vertex graph with $\delta(G) \geq n/2$, then $G$ contains a Hamilton cycle.

Equivalently, we can state Dirac’s theorem in the language of resilience.

Theorem (Dirac)

The complete graph $K_n$ is $\lfloor n/2 \rfloor$-resilient with respect to Hamiltonicity.
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Hamiltonicity in the binominal random graph $G_{n,p}$ is well studied.

**Theorem (Pósa, 1976; Koršunov, 1976)**

For $p \gg \log n/n$ we have that $G_{n,p}$ contains a Hamiltonian cycle asymptotically almost surely.

**Note:** $p \ll \log n/n \implies G_{n,p}$ will contain isolated vertices a.a.s.
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Dirac’s theorem for random graphs

**Theorem (Lee and Sudakov, 2012)**

*For* $p \gg \log n / n$, *the random graph* $G_{n,p}$ *is a.a.s. $(1/2 - o(1))np$-resilient with respect to Hamiltonicity.*
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For $p \gg \log n/n$, the random graph $G_{n,p}$ is a.a.s. $(1/2 - o(1))np$-resilient with respect to Hamiltonicity.

Note that the above threshold is tight: if we could delete anymore edges we could disconnect the graph.
Dirac’s theorem for random regular graphs

We generate a random regular graph via the model $G_{n,d}$ by choosing a graph uniformly at random among the set of $d$-regular graphs on $n$ vertices.

The following result follows from the work of Robinson and Wormald; Cooper, Frieze, Reed; Krivelevich, Sudakov, Vu, Wormald.

Theorem ($G_{n,d}$ is Hamiltonian)

For all $3 \leq d \leq n - 1$ we have that $G_{n,d}$ is Hamiltonian a.a.s.

Theorem (Ben-Shimon, Krivelevich and Sudakov, 2011)

For every $\varepsilon > 0$ and $d$ sufficiently large, a.a.s. $G_{n,d}$ is $(1 - \varepsilon)\frac{d}{6}$-resilient with respect to Hamiltonicity.

They conjectured that the true value should be closer to $\frac{d}{2}$. 

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Theorem (Condon, Espuny-Díaz, Girão, Kühn and Osthus, 2019+)

For every $\varepsilon > 0$ there exists $D$ such that, for every $d > D$, the random graph $G_{n,d}$ is a.a.s. $(1/2 - \varepsilon)d$-resilient with respect to Hamiltonicity.

Theorem (Condon, Espuny-Díaz, Girão, Kühn and Osthus, 2019+)

For any odd $d > 2$, the random graph $G_{n,d}$ is not a.a.s. $\left(\frac{d - 1}{2}\right)$-resilient with respect to Hamiltonicity.
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For any odd $d > 2$, the random graph $G_{n,d}$ is not a.a.s. $(d - 1)/2$-resilient with respect to Hamiltonicity.
Beyond Dirac

Theorem (Pósa, 1962)

Let $G$ have degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ such that $d_i \geq i + 1$ for all $i < n/2$. Then, $G$ is Hamiltonian.

Theorem (Chvátal, 1972)

Let $G$ have degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ such that, for all $i < n/2$, we have that $d_i \geq i + 1$ or $d_n - i \geq n - i$. Then, $G$ is Hamiltonian.

Question: Do Pósa's and Chvátal's results have corresponding analogues in $G_{n,p}$, like Dirac's result?

Answer: YES for Pósa, NO for Chvátal.
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Pósa’s theorem for random graphs.

**Theorem (Condon, Espuny Díaz, Kim, Kühn, Osthus, '18+)**

For every $\varepsilon > 0$, there exists $C > 0$ such that, for $p \geq C\log n/n$, a.a.s. every subgraph $G$ of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ with $d_i \geq (i + \varepsilon n)p$ for all $i < n/2$ is Hamiltonian.
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There exist counterexamples to Chvátal for random graphs.

**Theorem (Condon, Espuny Díaz, Kim, Kühn, Osthus, ’18+)**

For $p \gg \log n / n$, a.a.s. there exist subgraphs $G$ of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ satisfying $d_i \geq (i + \varepsilon n)p$ or $d_{n-i} \geq (n-i + \varepsilon n)p$ for all $i < n/2$ which are not Hamiltonian.

In fact, there exist subgraphs not containing a perfect matching.
Proof ideas: Dirac in $G_{n,d}$

Consider $G = G_{n,d}$. 

Definition (3-expander) An $n$-vertex graph $G$ is called a 3-expander if it is connected and, for every $S \subseteq \left[\frac{n}{400}\right]$ with $|S| \leq \frac{n}{400}$, we have $|N_G(S)| \geq 3|S|$. 

We show there exists a 'sparse' subgraph $R \subseteq G'$ which is a 3-expander.
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- We show there exists a ‘sparse’ subgraph $R \subseteq G'$ which is a 3-expander.
Proof outline: finding boosters

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Proof outline: finding boosters

- We consider longest paths in $R$.
- By a theorem of Pósa a 3-expander has many of such paths, with different endpoints.

$\implies$ there is a ‘large’ set edges whose inclusion would make $R$ Hamiltonian, or increase the length of a longest path.

In fact, we consider ‘booster’ pairs of edges, which have the same effect.
By passing from $R$ to $G'$ we argue that some of these booster pairs must exist.
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We add these edges to $R$ to make it Hamiltonian or else to increase the length of a longest path.
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We iterate this process at most $n$ times.
Open problems: $G_{n,p}$

Shifted Chvátal resilience.

Conjecture (Condon, Espuny Díaz, Kim, Kühn, Osthus, ’18+)

For $p \gg \log n/n$, a.a.s. every subgraph $G$ of $G_{n,p}$ with degree sequence $(d_1, \ldots, d_n)$ satisfying $d_i \geq (i + \varepsilon n)p$ or $d_{n-i-\varepsilon n} \geq (n - i + \varepsilon n)p$ for all $i < n/2$ is Hamiltonian.
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The conjecture holds for perfect matchings.

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Can we obtain bounds on the resilience for small $d$?
Open problems: $G_{n,d}$

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**Question**

Given any fixed even $d$, determine whether the graph $G_{n,d}$ is a.a.s. $(d/2 - 1)$-resilient with respect to Hamiltonicity.
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What is the likely resilience of $G_{n,4}$ with respect to Hamiltonicity? Is a graph obtained from $G_{n,4}$ by removing any matching a.a.s. Hamiltonian?