CONTINUOUS TRUTH I
Non-constructive Objects

Michael P. Fourman

Department of Mathematics
Columbia University
New York, N.Y. 10027
U.S.A.

Department of Pure Mathematics
University of Sydney
N.S.W. 2006
Australia

We give a general theory of the logic of potentially
infinite objects, derived from a theory of meaning for
statements concerning these objects. The paper has two
main parts which may be read independently but are
intended to complement each other. The first part is
essentially philosophical. In it, we discuss the theory
of meaning. We believe that even the staunchest realist
must view potential infinities operationally. The second
part is formal. In it, we consider the interpretation of
logic in the gros topos of sheaves over the category of
separable locales equipped with the open cover topology.
We show that general principles of continuity, local
choice and local compactness hold for these models. We
conclude with a brief discussion of the philosophical
significance of our formal results. They allow us to
reconcile our explanation of meaning with the "equivalence
thesis", that ⌜snow is white is true⌝ iff snow is white.

PROLEGOMENON

Classical mathematics is based on a platonic view of mathematical objects. The
meanings of mathematical statements are determined truth-functionally. This
Fregean explanation of meaning justifies classical logic. The deficiencies of
such a view are amply discussed by Dummett [1978].

A constructive mathematician rejects the completed infinities of classical mathe-
matics. For him, the *objects* of mathematics are essentially *finite*. The meaning
of quantification over infinite domains is given operationally in terms of a
theory of constructions. The resulting logic includes Heyting's predicate cal-
culus and other principles (e.g. choice principles).

As Dummett has stressed, one task of any philosophy of mathematics is to explain
the applicability of mathematics. The potential infinities of experience exceed
the finite objects of the strict constructivist. They demand a mathematics of in-
finite *objects*. Naive abstraction leads to the ideal infinite objects of class-
ical mathematics. This idealisation has enjoyed remarkable success. However,
the *meaning* of statements of classical mathematics remains problematic.

Brouwer [1981] introduced to mathematics potentially infinite objects such as free-
choice sequences. Consideration of these justified, for Brouwer, intuitionistic
logic, including various *choice and continuity principles*. We shall consider a
general notion of non-constructive object. For us, to present such a notion is to
give a *theory of meaning* for statements involving non-constructive objects.

Our non-constructive objects are not the platonic ideal objects of classical
mathematics nor the finitary objects of pure constructivism. They are potentially

infinite objects related to the lawless sequences of Kreisel [1968] and to Brouwer's free-choice sequences (Troelstra [1977]). The meanings of statements about these objects cannot be given in terms of truth conditions (as for classical platonist mathematics) or in terms of constructions (as for naive constructivism). The essence of these non-constructive objects lies in their infinite character. They are not, in general, totally grasped. They are given in terms of partial data which may later be refined. Meaning for statements about non-constructive objects is given by saying what data justifies a given assertion.

To describe a particular notion of non-constructive object is to describe the type of data on which it is based. We consider various such notions. Each conception of data gives an explanation of meaning which extends the range of meaningful statements and may be viewed as introducing new objects in that it ascribes meaning to new forms of quantification. In fact for each type of data we introduce a concrete representation of the non-constructive objects based on it.

Such a project is not novel: Beth [1947] introduced his models to provide just such an explanation of meaning for choice sequences. Our models generalise Beth's. Dummett [1977] makes a lengthy critique of the view that the intended meanings of of the logical constants are faithfully represented on Beth trees. Since our models generalise Beth's they appear *prima facie* to be susceptible to the same criticisms. However, Dummett's remarks on the (non)-consonance of the intended meanings of the connectives with their interpretation in Beth trees are directed at a different problem from the one we address. Dummett appears to have overlooked the possibility of separating the problem of explaining the constructive meaning of statements concerning lawlike objects from that of explaining the intuitionistic meaning of statements concerning choice sequences. Although we know of no satisfactory explanation of constructive truth (in particular, we agree with Dummett that Beth models do not give one), such a separation appears natural. It is possible to conceive of constructive truth independently of choice sequences. Given such a conception, Beth models provide *an* account of the introduction of non-lawlike objects. It is this type of account we have generalised. By way of example we now consider two notions of data closely related to Beth models. They both arise from the same informal picture.

Imagine receiving from Mars an infinite sequence α of natural numbers. The picture is of a ticker-tape which produces an indefinitely continued finite initial segment a of the sequence α. (We write $\alpha \in a$ to mean that a is an initial segment of α.) We want to examine the consequences of treating such undetermined sequences seriously as sequences. (Later we shall introduce more interesting examples.)

A naive view of this example considers the stages by which information arises: at any stage, the possible future data is represented by the collection $N^{<N}$ of all finite sequences. At the next stage one of the sequences <n> (where $n \in N$) must appear. Not only do we not yet know which of these possibilities will occur, it is not yet determined which will occur. On the basis of this data we may construct many sequences. The simplest of these, α is given by transcribing the data as it arrives. Thus on the basis of data b ≤ a, we are justified in asserting that a is an initial segment of α. We write this $b \Vdash \alpha \in a$. (We order sequences by setting b ≤ a if a is an initial segment of b since then b allows fewer possibilities for α.) Another sequence β is given by first writing down a fixed finite sequence b and then continuing with the incoming data. Schematically, $a \Vdash \beta \in b*a$ where * denotes concatenation, and hence "obviously", for any c ≤ a, we have $c \Vdash \beta \in b*a$. We want to make all such "obvious" assumptions about the nature of justification explicit; so we give its two basic structural properties. Justification should be

persistent $$\frac{a \Vdash \phi}{a*b \Vdash \phi}$$

and

inductive
$$\frac{a^* \ <n> \ \| \!\!-\ \phi \quad \text{for all } n \in N}{a \ \| \!\!-\ \phi}$$

Persistence reflects the idea that knowledge, once justified, is secure. The inductive clause comes from reflection on the infinite character of α. Given $\alpha \in a$, the collection $\{a^* \ <n> \ | \ n \in N\}$ *covers* all possibilities for future data.

In general, if we stipulate $b \| \!\!-\ \phi$ for $b \in B \subseteq N^{<N}$, the principles of induction and persistence will entail other instances $a \| \!\!-\ \phi$. We call the set $\{a \ | \ a \| \!\!-\ \phi\}$ the *inductive closure* of B and say B covers each such a.

To describe a particular non-constructive object γ is to say what information about it is justified on the basis of each possible item of future data. For example, any monotone function $g: N^{<N} \to N^{<N}$ (that is $a \leq b$ implies $g(a) \leq g(b)$) gives a non-constructive sequence γ with $a \| \!\!-\ \gamma \in g(a)$ (together with the consequences entailed by the principles of persistence and induction). Monotonicity here embodies the requirement that we should *accumulate* information on γ. More generally, we can view any poset P as an abstract representation of data about some type of object. Then a monotone map $N^{<N} \to P$ will represent a non-constructive object of this type.

We do not wish to introduce technicalities here. Later we shall give a definition of $a \| \!\!-\ \phi$, for first-order ϕ, by induction on the structure of ϕ. For the moment we just remark that such a definition of $a \| \!\!-\ \phi$ can be given and that the interpretations of the connectives are completely determined, in the context of our requirements on justification, by requiring that the rules of positive logic be valid. In our present case this would amount to giving Beth's semantics for intuitionistic logic with a slightly modified notion of "bar". Beth's semantics are well known to be equivalent to the topological interpretation over Baire space exploited by Scott [1968] and Moschovakis [1973]; our modification replaces Baire space by formal Baire space (Fourman and Grayson [1982]).

We now return to our picture of the ticker-tape. What we have done is to give a representation of the *subjective* experience of receiving word from Mars, an external view of how the world will look when data a appears on the ticker-tape. This view is dependent on the contingencies of what data is available. But mathematics should be timeless and absolute. This requirement appears to exclude consideration of potentially infinite objects. We now attempt to resolve this contradiction.

Picture a room with a ticker-tape, A and various sequences in progress, · α, β, γ described above for example. Now suppose that on the tape we have the finite sequence b. We have duly noted that $\alpha \in b$, $\beta \in b*b$, $\gamma \in g(b)$. Consider now another room A' with a ticker-tape which, as yet, is blank and three non-constructive objects defined by

$$c \| \!\!-\ \alpha' \in b*c$$
$$c \| \!\!-\ \beta' \in b*b*c$$
$$c \| \!\!-\ \gamma' \in g(b*c) \ .$$

In this room on the basis of no data we can already note that $\alpha' \in b$, $\beta' \in b*b$, $\gamma' \in g(b)$. Furthermore data b*c arriving in room A will always have the same consequences for α, β, γ as data c arriving in room A' has for α', β', γ'. In fact the mathematics (and logic) of the two rooms, A with data b and A' with no data, · should be the same. We want to add this to our formal treatment. This is done by regarding incoming data not as changing the world but rather as effecting a transformation which changes our view of the world. We consider not a particular ticker-tape but rather the uses which could be made of such an indeterminate sequence to generate non-constructive objects. Data just becomes a way of trans-

forming one such process into another, generally less free: its restriction.

We give a general definition of this transformation as follows:

$$c \Vdash \phi(\delta \restriction b) \quad \text{iff} \quad b*c \Vdash \phi(\delta)$$

(Where δ is a non-constructive object given by stipulating what data justifies $\phi(\delta)$ for various ϕ.) For example, $\alpha \restriction b = \alpha' = \beta$; $\beta \restriction b = \beta'$; $\gamma \restriction b = \gamma'$.

This change of viewpoint amounts formally to a change in our representation of data. Formerly we considered the partially ordered set or tree $N^{<N}$ as representing various possible states of information. Incoming data changes the world in that it places us in a new state. Now we consider $N^{<N}$ as a collection of transformations which act to change our view of the world.

Formally it is convenient to represent the data as the *monoid* of finite sequences under concatenation; if $g: N^{<N} \to P$ represents a non-constructive object γ then $\gamma | b$ is represented by $g \circ b$ where $b: a \mapsto b*a$ acts by left concatenation. The notion of justification is to be stable under such a change of perspective:

$$a \Vdash \phi \restriction b \quad \text{iff} \quad b*a \Vdash \phi$$

(where $\restriction b$ is applied to the non-constructive parameters of ϕ.)

We now consider examples of a more general type of non-constructive object intended to represent potential infinities of experience. We base our description, for the sake of exposition, on a view of classical experimental physics which we ascribe to the nineteenth century. Briefly it runs as follows: Physics is based on measurement. Experiments determine values of parameters a to a certain degree of precision. Generally some error is inevitable but it may in principle be made arbitrarily small (this is the assumption which leads us to label this as a nineteenth century notion).

Now, we refuse to admit the classical ascription of actual values to these parameters. At first consideration this may appear churlish. There is an apparent difference between a sequence determined only by the free will of a Martian and a physical value. We leave aside the question of whether this is an actual difference because this question misses the point. The point is to ask, "How can we assign meaning to statements concerning such quantities, in particular how should we understand quantification over such quantities?" Our refusal amounts to denying the coherence of any explanation based on the assumption that every sentence has a determinate truth value, either *true* or *false*. We refer to Dummett for elaboration of this point.

The possible results of experiments are concrete however. In general we may find by experiment that a $\in$ U. These possible U form a poset $\mathbb{P}$. The conceit which assumes that arbitrary refinement of our methods is in principle possible is represented by saying that the $V_i \le U$ representing a certain degree of refinement *cover* U.

For example, measurement of a quantity classically represented by a real parameter could be represented by taking for $\mathbb{P}$ the poset of rational open intervals, with the stipulation that for each $\epsilon > 0$ an open interval U is covered by the collection of all subintervals of length $\le \epsilon$, also that the collection of all proper subintervals of U covers U.

In general then we consider a poset $\mathbb{P}$ of possible outcomes for an experiment. We ask, as a technical convenience, that if p and q represent *a priori* compatible results (i.e. if there is an r with $r \le p$ and $r \le q$) then we can consider the outcome which consists just of getting these two results (i.e. we have an infinium $p \wedge q \in \mathbb{P}$). We also consider no information as a possible result (i.e. $\mathbb{P}$ has a

top element T). The poset $\mathbb{P}$ is equipped with a notion of covering family. We
demand that this be

reflective   {p} covers p

stable   If U covers p and q ≤ p then {q ∧ w | w ∈ U} covers q

monotone   If V ⊇ U covers p then V covers p.

The notion of a covering family is crucial to our explanation of meaning for in-
complete objects. It formalises the sense in which they are potentially infinite.
We avoid the metaphor of Wright [1981] which represents such a covering family as
embodying the recognition that the state of information is capable of effective
enlargement to one of type a* <n> because it seems to leave open to us the choice
of not performing this enlargement. The idea we have is to introduce consideration
of a particular type of incomplete object by specifying the type of data which
generates it. This specification includes a notion of covering family. Differ-
ences over which is the proper collection of covering families do not affect the
basic conception but merely lead to different types of data.

We are not as mathematicians or logicians interested in the result of a particular
experiment. Rather, we are interested in those properties which would remain in-
variant no matter what the outcome or methodology of a particular experiment. It
is not the result but the uses to which the result might be put in defining mathe-
matical quantities which interest us. Were the temperature scale non-linear, or
the time scale given by the unequal time of the sun, physics would be different
(it was). But mathematics and logic should be immune to such vagaries.

Our solution is similar to that we employed in giving an objective view of open
data. The possibility we envisage is that of a change of scale which in some
sense refines our possibilities for measurement. The measurements of the old con-
text should be meaningful in the new one but the new one may afford finer distinc-
tions. To describe such a change of scale is to say which new observations q ∈ $\mathbb{Q}$
are to be viewed as refining an old observation p ∈ $\mathbb{P}$. We write this relation
q ≤ f*(p) and demand that it be

monotone $$\frac{q' \le q \le f^*(p) \qquad p \le p'}{q' \le f^*(p')}$$

multiplicative $$\frac{q \le f^*(p) \qquad q \le f^*(p')}{q \le f^*(p \wedge p')}$$

continuous $$\frac{p_i \mid i \in I \;\; \text{covers } p \qquad r \le f^*(p)}{\{q \mid q \le f^*(p_i) \;\text{some}\; i \in I\} \;\text{covers}\; r} \;\; .$$

The motivation for the first two is clear. Continuity may be viewed as the re-
quirement that a previous conviction that a certain family covers, cannot be over-
turned. The change of viewpoint induced by such a transformation f is given by

$$\frac{p \Vdash \phi(\delta) \qquad q \le f^*(p)}{q \Vdash \phi(\delta \restriction f)} \qquad .$$

Mathematically, our notion of data gives a presentation of a *locale*. Change of
scale is represented by a continuous function between locales. Abstractly we
write such a change f: Y → X.

We now consider a supplement to our notion of justification. Suppose, we consider,
that consideration of a particular type of data would justify φ, then φ is justi-
fied. This is the reflection on which our whole project is based: that we can
justify talk of incomplete objects by reflecting on *hypothetical* indefinitely
continued processes.

We shall formulate this by saying that if f: Y → X represents the introduction of new distinctions *independent* of those represented by X then

$$\frac{Y \Vdash \phi \uparrow f}{X \Vdash \phi}$$

or that such an f is a *cover*.

Our final problem of formalisation is to characterise the introduction of *independent* data. A simple example is, given $\mathbb{P}$ and $\mathbb{Q}$ with notions of covering, to consider $\mathbb{P} \times \mathbb{Q}$ the product poset with coverings

$$\{<p_i,q> \mid i \in I\} \text{ covers } p,q \text{ when } \{p_i \mid i \in I\} \text{ covers } p$$
$$\{<p,q_i> \mid i \in I\} \text{ covers } p,q \text{ when } q_i \mid i \in I\} \text{ covers } q.$$

The projection given by $<p,q> \leq \pi^*(p')$ iff $p \leq p'$ represents the introduction of data of type $\mathbb{Q}$ independently of the data $\mathbb{P}$ under consideration. We shall require that all such projections be covers.

In general there are two conditions we require to view a change of scale as the introduction of independent data. The first is obvious: no new covers should be introduced between existing observations.

$$\frac{q_i \leq f^*(r_i) \qquad \{q_i \mid i \in I\} \text{ covers each } q \leq f^*(r)}{\{r_i \mid i \in I\} \text{ covers } r} \quad .$$

The second is subtle: no new conditional relationships should be introduced between existing observations. We explain: if $w \in \mathbb{Q}$ is such that

$$\frac{r \leq f^*(p) \qquad r \leq w}{r \leq f^*(q)}$$

(we view w as establishing a conditional relationship between $f^*(p)$ and $f^*(q)$), we demand that $w \leq f^*(s)$ for some $s \in \mathbb{P}$ such that

$$\frac{r \leq p \qquad r \leq s}{r \leq q}$$

(that the relationship be already established in $\mathbb{P}$). Technically, these requirements amount to demanding that the continuous map f: Y → X be a surjection and that it be open. The structure of data we have arrived at may be viewed as the category of locales equipped with the topology of covering by open maps.

Before turning to a formal examination of the interpretation of logic over this site, we sum up our intentions.

We introduce non-constructive objects by explaining the meanings of the connectives for statements concerning them. This is *not* a matter of characterising a domain of quantification. We have to explain the connectives anew in terms of the way such an object is given to us. Moreover, it is not sufficient to merely paraphrase the new quantifiers $\forall \alpha$ and $\exists \alpha$. Such a paraphrase entails a revision of the interpretations of → and ∨.

Our aim is to show that it is possible to derive rigorously properties of various domains of incomplete objects by giving a formal representation of the data which presents them as a site. We consider that the passage from an informal notion of data to the corresponding site is simple and natural. (Indeed, for us, to have a clear conception of a type of data is to be able to describe the corresponding site.) Once this passage is made, the derivation of properties (choice and continuity principles, for example) is a mathematical matter. Our hope in presenting these models is Leibnitzian: to eliminate further discussion of the justification of such principles by reducing the matter to calculation.

In our paper "Notions of Choice Sequence" [1982] we presented various notions of

choice sequence, including ones satisfying the axioms of LS and CS, with the same
purpose. Unfortunately, as the literature on choice sequences makes clear, clarity
is in the eye of the beholder. Hence the present attempt at a more careful explan-
ation of our informal notion of non-constructive object and its formalisation.

CONTINUOUS TRUTH

We start with a concrete presentation of the interpretation of higher-order logic
in a Grothendieck topos. This material (§§1-3) is well-known to cognoscenti
(tautologously), but is otherwise accessible only through a study of scattered
references. We give some of these sources but make no systematic attempt at a
complete list. Many important and historically significant contributions are not
mentioned. Our account is fuller than is logically necessary for the sequel in
order to point out some connections between different approaches. It is not, how-
ever, exhaustive.

§1   Frames and Locales

1.1 Definition.   A *frame* is a complete lattice with finite ∧ distributive over
arbitrary $\bigvee$. Frame morphisms, "*and-or maps*", are maps preserving these operations;
T,∧,$\bigvee$.

1.2 Example.   The lattice $O(X)$ of open subsets of a topological space is a frame.
If f: Y → X is a continuous map then the *inverse image* f*: $O(X)$ → $O(Y)$ is an
∧,$\bigvee$-map.

1.3 Definition. The category of *locales* or generalised *spaces* is the dual of the
category of frames. We call the morphisms *continuous maps* f: Y → X and write
f*: $O(X)$ → $O(Y)$ for the corresponding inverse image maps between the *frames of
opens* of X and Y (as in the topological case).

Example 1.2 gives a functor β: Top → Loc from topological spaces and continuous
maps to locales.

1.4 Discrete spaces.
Spatially $P(A)$ corresponds to the discrete topology on A. An example is the one-
point space with $O(\mathbb{1}) = P(\mathbb{1})$.

1.5 Definition.   A topological space X is *sober* iff Top$[\mathbb{1},X] \cong$ Loc$[\mathbb{1},\beta X]$.

1.6 Lemma.   On the full subcategory of sober spaces β is full and faithful.

We tacitly restrict our attention to sober spaces and henceforth omit mention of β.
We view locales as *generalised* spaces. (The relationship between Loc and Top is
better expressed in terms of the right adjoint, pt: Loc → Top, to β.)

1.7 Subspaces. Quotient maps of frames induce congruences: if f*: $O(X)$ → $O(Y)$ is
surjective let p ≈ q iff f*p = f*q. Each congruence class [p] has a canonical
representative jp = $\bigvee\{q \mid p \approx q\}$. The maps j: $O(X)$ → $O(X)$ arising in this way are

monotone $p \leq jp$

idempotent $j^2 = j$

multiplicative $j(p \wedge q) = jp \wedge jq$.

Such maps are called *nuclei*. The quotient may be identified as the image (or
fixed points) of j. The quotients of $O(X)$ are isomorphic (as posets) with the
nuclei on $O(X)$. Spatially we view these quotients as giving rise to subspaces
of X.

1.8 Surjections.   Dually, we view injective inverse image maps as giving rise to surjections of spaces.

1.9 Right adjoints. Each frame map f* has a right adjoint f_*, *direct image*, given by

$$f_* p = \bigvee \{q \mid f^* q \leq p\}.$$

The map $q \longmapsto p \wedge q$ has a right adjoint $r \longmapsto p \rightarrow r$ defined by

$$p \wedge q \leq r \qquad \text{iff} \qquad q \leq p \rightarrow r$$

where $p \rightarrow r = \bigvee \{q \mid p \wedge q \leq r\}$. Thus frames are complete Heyting algebras (but the morphisms are different).

1.10 Definition.   A map of spaces $f: X \rightarrow Y$ is *open* if the inverse image map $f^*: O(Y) \rightarrow O(X)$ has a left adjoint $\exists_f: O(X) \rightarrow O(Y)$ commuting with $\wedge$:

$$\exists_f (f^*(y) \wedge x) = y \wedge \exists_f(x)$$

or, equivalently, if f* preserves $\rightarrow$.

1.11 Proposition.   The category of locales is complete and cocomplete. Open surjections are stable (under pull-back).

The theory of locales is developed extensively by Joyal and Tierney [1982]. Johnstone [1982] uses locales systematically and has a comprehensive bibliography.

§2   Sites and Sheaves

2.1 Definitions.   Let $\mathbb{C}$ be a small category. A *crible* K of $A \in |\mathbb{C}|$ is a sub-functor of the representable functor $A \in S^{\mathbb{C}^{op}}$: that is, for each $B \in |\mathbb{C}|$ a set $K(B) \subseteq [B,A]$, stable under composition; for each $f \in K(B)$ and $g: C \rightarrow B$ in C, the composite $f \circ g \in K(C)$.

2.2 Lemma. The cribles of A form a frame, $P(A)$.
If $f: B \rightarrow A$ in $\mathbb{C}$ we have an *inverse image map* $f^*: P(A) \rightarrow P(B)$ given by $f^* K = \{g \mid f \circ g \in K\}$ for $K \in P(A)$. By abuse we write $f: \check{B} \rightarrow \check{A}$ for the corresponding continuous map. This map is open.

2.3 Definition.   (Lawvere-Tierney) A Grothendieck topology j on $\mathbb{C}$ is a family of nuclei $j_A: P(A) \rightarrow P(A)$, natural in A: that is $f^* \circ j_A = j_B \circ f^*$ for $f: B \rightarrow A$.

2.4 Lemma.   If j is a Grothendieck topology on $\mathbb{C}$, the quotient frames $\Omega(A)$ have induced inverse image maps $f^*: \Omega(A) \rightarrow \Omega(B)$ and the corresponding map of locales, which we write $f: B^j \rightarrow A^j$, is open.

2.5 Definitions.   A *pretopology* J on $\mathbb{C}$ is a family $J(A) \subseteq P(A)$ for each $A \in \mathbb{C}$ which is

reflexive $A \in J(A)$

multiplicative $$\frac{K \in J(A) \qquad L \in J(A)}{K \cap L \in J(A)}$$

stable $$\frac{K \in J(A) \qquad f: B \rightarrow A}{f^* K \in J(B)}.$$

(For example, let $K \in J(A)$ iff $jK = T$.)

A crible $K \in P(A)$ is inductively *closed* for J iff

$$\frac{f: B \to A \qquad f^*K \in J(B)}{f \in K} \quad .$$

As A is closed and an intersection of closed cribles is closed, each crible
$K \in P(A)$ has a *closure* $j_A K$. This gives a topology j on $\mathbb{C}$. We say K *inductively*
covers A iff jK = A, and write this $K \in \bar{J}(A)$.

2.6 Examples. (1) Let $O(X)$ be a frame viewed as a poset viewed as a category.
Let $K \in J(A)$ iff $\bigvee K = A$. (Identifying B → A with its domain.) Then $\Omega(X) \cong O(X)$.
(2) Let $\mathbb{C}$ be a small category of locales closed under finite limits and open
inclusions. Let $K \in J(A)$ iff K contains some family $\{f_i: B_i \to A | i \in I\}$ of *open*
maps such that $\bigvee \exists_{f_i}(B_i) = A$. The crible generated by each open inclusion U ↪ A
is closed for this topology. This assignment gives an $\wedge; \bigvee$ map $r^*: O(A) \to \Omega(A)$,
spatially a surjection $r: \underline{A}^j \to A$. Each closed crible contains a largest open
inclusion. This assignment gives an $\wedge \bigvee$ map $i^*: \Omega(A) \to O(A)$, spatially we have
an inclusion $i: A \hookrightarrow \underline{A}^j$. Furthermore, $r \circ i = id_A$ and $r^*i^* \leq id_{\Omega(A)}$ so A is an
adjoint retract of $\underline{A}^j$ (Fourman [1982]).

2.7 Definition. A *presheaf* on $\mathbb{C}$ is a functor X: $\mathbb{C}^{op} \to$ Sets. If f: B → A $\in \mathbb{C}$ and
$a \in X(A)$ we use the notation $a\!\restriction\!f$, "a restricted along f", for $X(f)(a) \in X(B)$.
Note that $a\!\restriction\!f\!\restriction\!g = a\!\restriction\!f \circ g$ and $a\!\restriction\!id = a$. The appropriate *morphisms between pre-*
sheaves F: Y → X are natural transformations, maps $F_A: Y(A) \to X(A)$ which commute
with restrictions $F_B(a\!\restriction\!f) = (F_A a)\!\restriction\!f$.

2.8 Examples.   (1) The representable functor [-,A], (or by abuse, justified by
Yoneda's lemma, A) is a presheaf. Restrictions are by composition $g\!\restriction\!f = g \circ f$.
Yoneda's lemma tells us that for any other presheaf, X, we have $X(A) \cong [A,X]$. In
particular the embedding $\mathbb{C} \to S^{\mathbb{C}^{op}}$ is full and faithful. Each crible $K \in P(A)$ is a
subpresheaf K ↣ A.
(2) P and Ω are presheaves with restrictions given by inverse images $K\!\restriction\!f = f^*(K)$.

2.9 Definition. A presheaf X on $\mathbb{C}$ is a *sheaf* for the (pre) topology J if when-
ever $K \in J(A)$, each natural transformation x: K → X has a *unique* extension along
K ↣ A. Equivalently, if $K \in J(A)$ and we have a family $x_f \in X(B)$ for f: B → A $\in$ K
such that $x_{gf} = x_f\!\restriction\!g$ for each g: C → B, there is a unique $x \in X(A)$ such that
$x_f = x\!\restriction\!f$ for each f $\in$ K. The category of sheaves and natural transformations is a
Grothendieck topos.

There is, as yet, no staisfactory introductory text on topos theory. The basic
references are SGA4, Wraith [1975], Johnstone [1977], Freyd [1972].

§3   Forcing over a site

Here we describe Joyal's presentation of interpretations in topoi in terms of a
notion of forcing. Let $\mathbb{C}$ and a (pre) topology J be fixed. The basic structures
we consider are diagrams of *presheaves* on $\mathbb{C}$. Each presheaf A interprets a *type*
or *sort* of variable. A morphism f: $A_1 \times \ldots \times A_n \to$ B interprets an n-ary oper-
ation. A subobject R ↣ $A_1 \times \ldots \times A_n$ interprets an n-ary relation.

3.1 Definitions.   Let L be a first-order language (possibly many-sorted) with
equality. An interpretation of L is given by assigning to each sort A of L a
presheaf A, to each operation F from $A_1,\ldots,A_n$ to B a natural transformation
F: $A_1 \times \ldots \times A_n \to$ B, and to each relation R on $A_1 \times \ldots \times A_n$ a subfunctor
R ↣ $A_1 \times \ldots \times A_n$. Given such an interpretation, for U $\in |\mathbb{C}|$ we let L_U be the

expansion of L obtained by adding constants of the appropriate sorts for the ele-
ments of $A(U)$. If $f: V \to U$ then for any term τ or formula ϕ of L_U we obtain a
term $\tau\!\upharpoonright\!f$ or formula $\phi\!\upharpoonright\!f$ of L_V by restricting any new constants which occur. Now
for $U \in |\mathbb{C}|$ we define for each closed term τ of sort A of L_U an *interpretation*
$[\![\tau]\!]_U \in A(U)$ by induction:

$$[\![c]\!]_U = c \quad \text{for} \ c \in A(U)$$

$$[\![F(\tau_1,\ldots,\tau_n)]\!]_U = F([\![\tau_1]\!],\ldots,[\![\tau_n]\!]) \ .$$

Note that $[\![\tau\!\upharpoonright\!f]\!]_V = [\![\tau]\!]_U\!\upharpoonright\!f$. Now we define inductively the relation, U *forces* ϕ,

$$U \Vdash \phi$$

for ϕ a sentence of L_U.

INDUCTIVE DEFINITION OF FORCING

(I)
$$\frac{V_f \Vdash \phi\!\upharpoonright\!f \quad \text{all} \ f: V_f \to K}{U \Vdash \phi} \qquad K \in J(U)$$

$(\wedge)^+$
$$\frac{U \Vdash \phi \qquad U \Vdash \psi}{U \Vdash \phi \wedge \psi}$$

$(\vee)^+$
$$\frac{U \Vdash \phi}{U \Vdash \phi \vee \psi} \qquad \frac{U \Vdash \psi}{U \Vdash \phi \vee \psi}$$

$(\exists)^+$
$$\frac{U \Vdash [c/x]}{U \Vdash \exists x.\phi}$$

$(\to)^+$
$$\frac{\text{for all} \ f: V \to U, \ \text{if} \ V \Vdash \phi\!\upharpoonright\!f \ \text{then} \ V \Vdash \psi\!\upharpoonright\!f}{V \Vdash \phi \to \psi}$$

$(\forall)^+$
$$\frac{\text{for all} \ f: V \to U, \ \text{for all} \ c \in A(U), \ V \Vdash \phi\!\upharpoonright\!f[c/x]}{U \Vdash \forall x.\phi}$$

$(=)^+$
$$\frac{[\![\tau_1]\!] = [\![\tau_2]\!]}{U \Vdash \tau_1 = \tau_2}$$

$(\text{Atomic})^+$
$$\frac{\langle[\![\tau_1]\!],\ldots,[\![\tau_n]\!]\rangle \in R}{U \Vdash R(\tau_1, \ldots, \tau_n)}$$

We now give some "derived rules" for forcing:

3.2 Lemma. Basic properties of forcing

(P)
$$\frac{U \Vdash \phi \qquad f: V \to U}{V \Vdash \phi\!\upharpoonright\!f}$$

$(\wedge)^-$
$$\frac{U \Vdash \phi \wedge \psi}{U \Vdash \phi \qquad U \Vdash \psi}$$

$(\vee)^-$
$$\frac{U \Vdash \phi \vee \psi}{\text{for each } f \text{ in some } K \in \mathfrak{J}(U) \text{ either } V_f \Vdash \phi \uparrow f \text{ or } V_f \Vdash \psi \uparrow f}$$

$(\exists)^-$
$$\frac{U \Vdash \exists x. \phi}{\text{for each } f \text{ in some } K \in \mathfrak{J}(U) \text{ we have } V_f \Vdash \phi[c/x] \text{ for some } c \in A(V_f)}$$

$(\rightarrow)^-$
$$\frac{V \Vdash \phi \rightarrow \psi}{\text{for all } f \colon V \rightarrow U \text{ if } V \Vdash \phi \uparrow f \text{ then } V \Vdash \psi \uparrow f}$$

$(\forall)^-$
$$\frac{V \Vdash \forall x. \phi}{\text{for all } f \colon V \rightarrow U \text{ and all } c \in A(U), \text{ we have } V \Vdash \phi \uparrow f[c/x]}$$

$(=)^-$
$$\frac{U \Vdash \tau_1 = \tau_2}{\text{for each } f \text{ in some } K \in \mathfrak{J}(U) \text{ we have } [\![\tau_1]\!]\uparrow f = [\![\tau_2]\!]\uparrow f}$$

$(\text{Atomic})^-$
$$\frac{U \Vdash R(\tau_1, \ldots, \tau_n)}{\text{for each } f \text{ in some } K \in \mathfrak{J}(U) \text{ we have } \langle [\![\tau_1]\!]\uparrow f, \ldots, [\![\tau_n]\!]\uparrow f \rangle \in R(V_f)}$$

Our presentation here is non-standard in that the definition of forcing is usually given by stipulating both positive and negative rules for each connective, (I) and (P) are then derived. The resulting relation is the same.

3.3 Definition. A sequent $\Gamma \Rightarrow \phi$ is *valid* (written $\Gamma \models \phi$) in the given interpretation iff
$$\frac{U \Vdash \psi[\xi(x)/x] \text{ all } \psi \in \Gamma}{U \Vdash \phi[\xi(x)/x]}$$
where ξ is an interpreation of the variables of L by elements of the appropriate $A(U)$.

3.4 Proposition. If each sort is interpreted by an *inhabited* presheaf (each $A(U)$ is inhabited) then the axioms and rules of Heyting's propositional calculus are valid for $\models$.

(Adaptations for domains which are not inhabited are discussed in Fourman [1977], Scott [1978], Joyal & Boileau [1981], Makkai & Reyes [1977].)

3.5 Definitions. A presheaf A is *separated* iff
$$\frac{U \Vdash a = b}{a = b} \qquad \text{for } a, b \in A(U).$$
A subobject $R \rightarrowtail A$ is *closed* iff
$$\frac{U \Vdash R(a)}{a \in R(U)} \qquad \text{for } a \in A(U).$$

A higher-order type-theory is merely a many-sorted first-order theory with some structure on the collection of sorts and certain distinguished operations and relations. One of the insights due to Lawvere and Tierney is that topoi have such higher-order structure. We consider languages where for any two sorts A and B we can form the *product* $A \times B$ with appropriate *pairing* and *projection* operations, the *function space* B^A with an *evaluation operation* $-(-)$, and also the power type

$P(A)$ with a *membership relation* ϵ. An interpretation is *standard* if all this structure is interpreted by the corresponding structure on $Sh(\mathbb{C})$.

3.6 Proposition.   In any standard interpretation the following schemata, which combine *comprehension* and *extensionality*, are valid.

$$x \in A \; \exists! y \in B.\phi(x,y) \leftrightarrow \exists! f \in B^A \; \forall x \in A.\phi(x,f(x))$$

$$\exists! z \in P(A) \; \forall x \in A \quad (x \in z \leftrightarrow \phi(x)). \quad \square$$

Thus power-types and function spaces behave as they should. The categorical characterisation of this higher-order structure in terms of adjoints is very simple, products are categorical products,

$$\frac{C \to B^A}{A \times C \to B} \qquad \frac{C \to P(A)}{Sub(A \times C)} \; .$$

We shall not describe this structure in general here. We shall be dealing primarily with sorts interpreted by representables. These are particularly simple to deal with because they have *generic elements*. A well-known consequence of this is the Yoneda Lemma:

$$F(U) \cong [U,F] \quad \text{for} \quad F \in |S^{\mathbb{C}^{op}}| \quad \text{and} \quad U \in |\mathbb{C}| \; .$$

We use this to calculate some examples of the higher-order structure. For this exercise, we suppose that $\mathbb{C}$ has finite products and that each representable functor is a sheaf.

3.7 Lemma.   If F is a sheaf and U,V are representable

(1) $F^U(U) \cong F(U \times C)$

with evaluation for u: $V \to U$ and $\xi \in F(U \times V)$ given by $\xi(u) = \xi\uparrow<u,V>$.

(2) $(PU)(V) = \Omega(U \times V)$ with $U \Vdash a \in R$ iff $R\uparrow<a,V> = V$.

Proof. $F^U(V) \cong [V,F^U] \cong [U \times V,F] \cong F(U \times V)$

$(PU)(V) \cong [V,PU] \cong Sub(U \times V) = \Omega(U \times V). \quad \square$

A logical counterpart to Yoneda's lemma is the following.

3.8 Lemma.   *Generic elements for representables.*
If U is representable then $V \Vdash \forall x \in U.\phi$ iff $V \times U \Vdash \phi\uparrow\pi_1[\pi_2/x]$.

Proof.   In one direction this is immediate from $(\forall)^-$. In the other, suppose f: $W \to V$ with a: $W \to U \in U(W)$ then $<f,a>: W \to V \times U$ and, by persistence, if $V \times U \Vdash \phi\uparrow\pi_1[\pi_2/x]$ then $W \Vdash \phi\uparrow f[a/x]$. So by $(\forall)^+$ we have $V \Vdash \forall x.\phi$. $\square$

We give an example of the use of generic elements in the simple case of a category of presheaves.

3.9   Proposition.   Choice holds for representables in categories of presheaves.
Proof.   Let U be a representable and suppose

$$V \Vdash \forall x \in U.\exists y \in F.\phi(x,y)$$

then

$$U \times V \Vdash \exists y \in F.\phi\uparrow\pi_2(\pi_1,y)$$

$$U \times V \Vdash \phi\uparrow\pi_2(\pi_1,\xi) \quad \text{for some} \; \xi \in F(U \times V)$$

regarding ξ as an element of $F^U(V)$ this gives
$$U \times V \Vdash \phi \dashv \pi_2(\pi_1, (\xi \dashv \pi_2)(\pi_1)) \ .$$
Since $\xi \dashv <\pi_1, \pi_2> = \xi.$ Thus
$$V \Vdash \forall x \in U . \phi(x, \xi(x)),$$
and so
$$V \Vdash \exists f . \forall x . \phi(x, f(x)). \qquad \square$$

From a category-theoretic viewpoint this result is well-known in the form, "Representables are internally projective".

§4   POINTS, LOCAL CHOICE, CONTINUITY

Now we let $\mathbb{C}$ be a category of locales closed under finite limits and open inclusions, equipped with the open cover topology, J. We write $\mathbb{E}$ for the topos $Sh(\mathbb{C}, J)$.

For each locale X we define an internal locale $\mathbb{X}$ by
$$\mathcal{O}(\mathbb{X})(U) \cong \mathcal{O}(X \times U^j).$$
This is generated internally by the basis given by
$$\mathcal{B}(\mathbb{X})(U) \cong \mathcal{O}(X \times U)$$
or even by the constant basis
$$\mathcal{B}_0(\mathbb{X})(U) = \mathcal{O}(X),$$
with the inclusions $\mathcal{B}_0(\mathbb{X}) \hookrightarrow \mathcal{B}(\mathbb{X}) \hookrightarrow \mathcal{O}(\mathbb{X})$ induced by the projections $X \times U^j \to X \times U \to X$. (In the terminology of Joyal & Tierney $\mathbb{X} = P^*(X)$.)

The internal space of points of $\mathbb{X}$ is given by
$$(pt\mathbb{X})(U) \cong C[U^j, X].$$
This is the space of $\mathbb{E}$-valued models of X. For $\alpha: U^j \to X$ and $W \in \mathcal{O}(X \times U^j)$
$$U \Vdash \alpha \in W \quad \text{iff} \quad <\alpha, U^j>^{-1}(W) = T.$$
For $\alpha: U^j \to X$ and $V \in \mathcal{O}(X \times U)$
$$U \Vdash \alpha \in V \quad \text{iff} \quad [(X \times r)_0 <\alpha, U^j>]^{-1} V = T$$
$$\text{iff} \quad [X \times \alpha \circ i]^{-1} V = T$$
since $i^*: \mathcal{O}(U^j) \to \mathcal{O}(U)$ reflects T.

4.1 **Proposition.**   For any $X \in |\mathbb{C}|$ the internal locale $\mathbb{X}$ has enough points.
Proof.   We must show for $W \in \mathcal{O}(X)$ that
$$\frac{U \Vdash K \text{ covers pt } W \qquad U \Vdash K \text{ is an inductive crible}}{U \Vdash W \in K} \ .$$

We assume the hypotheses, and let
$$\mathbb{K} = \{W_i \times U_i \mid U_i \Vdash W_i \in K \dashv U_i\}$$

Clearly $\mathbb{K}$ is a doubly inductive crible of $\mathcal{O}(W) \times \mathcal{O}(U)$, that is an open of $W \times U$. We show that $\mathbb{K}$ covers $W \times U$ that is that $W \times U \in K$, which is evidently sufficient since then $U \Vdash W \in K$.

By persistence
$$W \times U \Vdash K \dashv \pi_2 \text{ covers } \pi_1$$

that is

$$W \times U \Vdash \exists V \in K \upharpoonright \pi_2 . \pi_1 \in V.$$

So

$$\mathbb{K}^* = \{W_i \times U_i \mid \text{for some } V_i \in O(W) \text{ we have } W_i \times U_i \Vdash V_i \in K \upharpoonright \pi_2 \wedge \pi_1 \in V_i\}$$

covers $W \times U$. [Because, if p: $X \to Y$ is an open surjection and $X \Vdash V \in K \mid p \wedge \theta \mid p \in V$ then $Y \Vdash V \in K \wedge \theta \in V$: the basic opens of $\mathbb{X}$ are constant and thus descend open surjections.] But now we claim $\mathbb{K}^* \subseteq \mathbb{K}$ because, by definition

$$W_i \times U_i \Vdash \pi_1 \in V_i \text{ iff } W_i \subseteq V_i$$

and, as projections are covers,

$$W_i \times U_i \Vdash V_i \in K \upharpoonright \pi_2 \text{ iff } U_i \Vdash V_i \in K \upharpoonright U_i. \quad \Box$$

Special cases of this are worthy of mention. When X is Baire space N^N, Cantor space 2^N, Dedekind reals R, to say that $\mathbb{X}$ has enough points is the internal statement of Bar induction, Fan theorem, Heine Borel theorem (respectively). For these cases it is sufficeint to take the topology on $\mathbb{C}$ generated by covering families of open *inclusions*: since each of these spaces X has a point the projections $X \times U \to X$ are covers for this topology. We call this topology the *open inclusion topology*.

We introduce some more general spaces. Let f: $X \to U$ in Loc. We consider the internal locale X/f defined at U by the basis $O(V)$ with all its standard covers. More properly for g: $W \to U$ we define

$$B(X/f) \upharpoonright g \cong O(g^*X)$$

given by pulling back f along g. Any commuting triangle
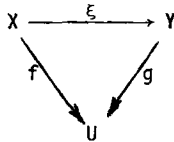

induces an internal map of locales

$$\xi: X/f \to Y/h$$

defined at U. Given by ξ^{-1} on basis elements, this clearly takes basic covers to covers. Furthermore, if ξ: $X \to Y$ is open (and surjective) then ξ: X/f $\to$ Y/h is open, since it suffices to define $\exists_\xi$ commuting with $\wedge$ on basis elements, (and surjective since if ξ: $X \to Y$ is an open surjection then so are all its pullbacks, so internally ξ^{-1} reflects basic open covers). These spaces include the spaces $\mathbb{X}$ we introduced earlier as

$$U \Vdash \mathbb{X} \cong (X \times U)/\pi .$$

We now specialise to the case where the objects of $\mathbb{C}$ are T_1. Then $U \hookrightarrow U^j$ induces an isomorphism

$$(p + \mathbb{X})(U) \cong C[U^j, X] \cong C[U, X]$$

so that X represents the functor pt$\mathbb{X}$. This happens in particular for the spaces N, N^N, 2^N, R and their basic opens (see Fourman [1983].) Furthermore, any element of pt(X/f) defined at U induces a commuting triangle
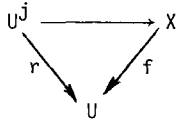
which for T_1 spaces
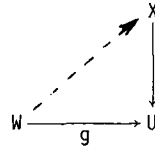X corresponds to a
section of g

So we have a presentation

$$pt(X/f) \!\uparrow\! g \qquad \text{correspond to}$$
commuting triangles

with restriction given by composition.

We extend our earlier lemma on generic elements:

<u>4.2 Lemma</u>.   If objects of $\mathbb{C}$ are T_1 then
$$U \Vdash \forall x \in pt(X/f).\phi$$

iff

$$X \Vdash (\phi\!\uparrow\!f)(id). \qquad \square$$

Of course these generalised representables can be defined internally in any
Grothendieck topos and this result holds.

<u>4.3 Proposition</u>.   If the objects of $\mathbb{C}$ are T_1 then for any $X \in |\mathbb{C}|$ and any
$A \in Sh(\mathbb{C})$
$$\vDash \forall x \in pt(\mathbb{X}).\exists a \in A.\phi(x,a)$$
$$\rightarrow \exists \text{ open cover } p: Z \longrightarrow\!\!\!\!\rightarrow \mathbb{X} \text{ and a function } f: pt\ Z \rightarrow A \text{ such that}$$
$$\forall x \in A.\forall z \in pt\ Z[pz = x \rightarrow \phi(x,f(z))].$$

<u>Proof</u>.   As $pt\ \mathbb{X}$ is representable,
$$U \Vdash \forall x\ \exists a\ \phi(x,a)$$

iff

$$X \times U \Vdash \exists a\ \phi\!\uparrow\!\pi_2(\pi_1,a)$$
iff for some open cover $p: Z \longrightarrow X \times U$
$$Z \Vdash \phi\!\uparrow\!\pi_2 \circ p(\pi_1 \circ p, \xi) \quad \text{for some } \xi \in A(Z)$$

iff

$$U \Vdash \forall z \in Z\ \phi(p(z),\xi(z)). \quad \square$$

We do not know under what conditions ξ descends to give a function defined on a
cover by open sets. We can ensure this by considering the open inclusion topology
on $\mathbb{C}$ in which case we obtain

$$\vDash \forall x \in pt\ \mathbb{X}.\exists a \in A.\phi(x,a)$$
$$\rightarrow \exists \text{ open cover } U_i \in O(X) \text{ and functions } f_i: U_i \rightarrow A \text{ such that}$$
$$\forall x \in U_i.\phi(x,f_i(x))\ .$$

We now consider continuity.

<u>4.4. Proposition</u>. If X,Y are T_1 then $\vDash \forall f: pt\ \mathbb{X} \rightarrow pt\ \mathbb{Y}$, f *is continuous*.

Proof.   If $U \Vdash f: pt\ \mathbb{X} \to pt\ \mathbb{Y}$ then f is represented by $\xi: X \times U \to Y$ in $\mathbb{C}$.   For $V \in \mathcal{O}(Y)$ a basic open of $\mathbb{Y}$, w: W $\to$ U and x: W $\to$ X we have

$$W \Vdash (\xi \upharpoonright w)(x) \in V$$

iff $$[\xi \circ <x,w>]^{-1}(V) = W$$

iff $$<x,w>^{-1} \xi^{-1}(V) = W$$

iff $$W \Vdash x \in [\xi^{-1}(V) \upharpoonright w]$$

regarding $\xi^{-1}V \in \mathcal{O}(X \times U)$ as an open of $\mathbb{X}$ defined at U.

Thus $U \Vdash \xi^{-1}(V)$ is open. $\square$

§5   Iteration

We return for a while to consideration of a general Grothendieck topos $\mathbb{E} = Sh(\mathbb{C}, J)$. We consider the internal category $\mathbb{C}$ in $\mathbb{E}$ given by

$$\underline{\mathbb{C}}(U) \cong \mathbb{C}/U$$

with restrictions given by pulling back.
[For those who worry about coherence (one should worry), we remark that a concrete category in $\mathbb{E}$ with an equivalent category of sections over U is given by considering V/f to be represented as the element S of (PV)(U) determined by

$$W \Vdash v \in S \upharpoonright g \quad iff \quad f \circ v = g .$$

So $\underline{\mathbb{C}}$ is an internal small full subcategory of $\mathbb{E}$ whose objects are subfunctors of representables.]
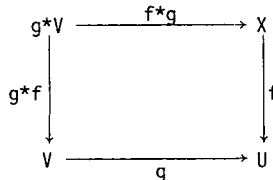We give $\underline{\mathbb{C}}$ a topology by letting
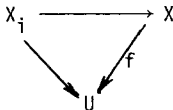
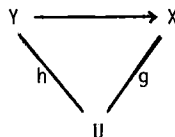cover X/f in $\underline{\mathbb{C}}$ if $X_i \to X$ cover X in $\underline{\mathbb{C}}$.

Now for $A \in |\mathbb{E}|$ we define $\underline{A} \in Sh_{\mathbb{E}}(C,J)$ by

$$U \Vdash \underline{A}(X/f) \quad \cong \quad A(X)$$

with restrictions for g: V $\to$ U given by restriction along f*g

and for ξ: Y/h $\to$ X/g in $\mathbb{C}/U$, by restriction along ξ

Any morphism A $\to$ B in $\mathbb{E}$ induces $\underline{A} \to \underline{B}$ in $Sh_{\mathbb{E}}(\underline{\mathbb{C}}, J)$.

For those who prefer global descriptions, we associate to $A \in |\mathbb{E}|$ (pseudo) functors

$$\mathbb{C}/U \to \mathbb{E}/U$$

natural in U (i.e. commuting with g* for g: V → U) as follows:

$$\begin{array}{c} X \\ \Big\downarrow f \\ U \end{array} \longmapsto \Pi_f \Delta A$$

where

$$\mathbb{E}/X \xrightarrow{\ \Pi f\ } \mathbb{E}/U$$

$$\Delta_X \searrow$$

$$\mathbb{E} \qquad .$$

For $Y \xrightarrow{\ \xi\ } X$ we have $\Pi_h \cong \Pi_f \Pi_\xi$ whence $\Pi_h \xi^* \Rightarrow \Pi_f$ (as $\xi^* \dashv \Pi_\xi$)

with h and f to U.

and $\Pi_h \Delta_Y \Rightarrow \Pi_f \Delta_X$ (as $\xi^* \Delta_X \cong \Delta_Y$).

This gives the required arrow $\Pi_h \Delta_Y A \to \Pi_f \Delta_X A$. What we obtain is an (internal) functor

$$\mathbb{E} \to \mathbb{E}^{\underset{\sim}{\mathbb{C}}^{op}} \qquad .$$

We shall show that this preserves first order logic.

Here we work concretely for the sake of computations. A simple but more abstract treatment will appear in Fourman and Kelly [1983]. We now consider a first-order language L with sorts for the objects of $\mathbb{E}$ and operations symbols for its morphisms. In fact to avoid size problems, we consider an arbitrary small fragment of such a language. We may consider L also as a language in $\mathbb{E}$ as a constant object (via Δ).

Working in $\mathbb{E}$ we consider the interpretation of L given by interpreting the sort A by $\underset{\sim}{A}$ and each operation f: A → B by the corresponding morphism $\underset{\sim}{A} \to \underset{\sim}{B}$.

5.1   Lemma.   For f: X → U and g: X → V

$$U \Vdash X/f \Vdash \phi \quad \text{iff} \quad V \Vdash X/g \Vdash \phi$$

Proof.   By induction, it suffices to show that if $U \Vdash X/f \Vdash^* \phi$ is defined to mean

$$V \Vdash X/g \Vdash \phi \quad \text{for all} \quad g: X \to V$$

then $\Vdash^*$ is closed under the rules of . As no rule decreases the complexity of φ we may assume that the result holds for subformulae of φ.

Only $(\to)^+$ and $(\forall)^+$ present any difficulties. We consider $(\to)^+$, and suppose the result for φ and ψ.

Suppose that for all ξ: W → U and all h: Z → ξ*X, if $W \Vdash Z/(\xi^* f \circ h) \Vdash^* \phi\{(f*\xi \circ h)$ then $W \Vdash Z/(\xi^* f \quad h) \Vdash^* \psi\{(f*\xi \circ h)$. Then if η: W' → V and h': Z → η*X are such that $W' \Vdash Z'/(\eta^* g \circ h') \Vdash \phi\{(g*\eta \circ h')$ then by induction hypothesis

$$U \Vdash Z'(f \circ g*\eta \circ h') \Vdash * \phi\uparrow(g*\eta \circ h')$$

whence (letting ξ = id and h = g*$\eta \circ$ h') we have

$$U \Vdash Z'/(f \circ g*\eta \circ h') \Vdash * \psi\uparrow(g*\eta \circ h')$$

in particular W' $\Vdash$ Z'($\eta*g \circ$ h') $\Vdash \psi\uparrow$(g*$\eta \circ$ h'). So V$\Vdash$ X/g$\Vdash \phi \to \psi$. The proof
for $\forall^+$ is similar.   □

5.2 Theorem.   For ϕ a formula of L with appropriate parameters

$$U \Vdash \text{"} X/f \Vdash \phi\text{"} \quad \text{iff} \quad X \Vdash \phi \quad .$$

Proof. Firstly, this is well formed: Parameters for ϕ at X/f are elements of
$\underline{A(X/f)}$ which are given as elements of A(X) and are thus parameters for ϕ at X.

We proceed by induction. That is, we show that if we define $\Vdash*$ internally by

$$U \Vdash X/f \Vdash * \phi \quad \text{iff} \quad X \Vdash \phi$$

then $\Vdash *$ is closed under the defining clauses of $\Vdash$ internally, (whence
U$\Vdash$ X/f$\Vdash \phi \Rightarrow$ X$\Vdash \phi$) and if we define $\Vdash^+$ by X $\Vdash^+ \phi$ iff U$\Vdash$ X/f$\Vdash \phi$ then $\Vdash^+$
is closed under the defining clauses of $\Vdash$ (whence X$\Vdash \phi \Rightarrow$ U$\Vdash$ X/f$\Vdash \phi$).

As the operations $\underset{\sim}{A} \to \underset{\sim}{B}$ are just those inherited from $\mathbb{E}$, terms are interpreted
alike in both contexts. Thus if $[\![\tau]\!]$ = $[\![\sigma]\!]$ then U$\Vdash [\![\tau]\!]$ = $[\![\sigma]\!]$, so $\Vdash^+$ is closed
under $(=)^+$ and if

$$U \Vdash [\![\tau]\!] = [\![\sigma]\!] \text{ then } U \Vdash \tau = \sigma,$$

so $\Vdash*$ is closed under $(=)^+$.

That $\Vdash^+$ and $\Vdash*$ are closed under $(\wedge)^+$, $(\vee)^+$, $(\exists)^+$ is trivial. For I, suppose
$X_i \Vdash^+ \phi\uparrow f_i$ for f_i: $X_i \to$ X in some cover of X then X$\Vdash X_i/f_i \Vdash \phi\uparrow f_i$ and by I
internally X$\Vdash$ X/id$\Vdash \phi$. In the contrary direction, suppose U$\Vdash X_i/g \circ f_i \Vdash * \phi\uparrow f_i$
for some cover of X as above. Then $X_i \Vdash \phi\uparrow f_i$ so X$\Vdash \phi$ that is U$\Vdash$ X/g$\Vdash \phi$. For
$(\to)^+$, first suppose that for all f: V $\to$ U if V $\Vdash^+ \phi\uparrow f$ then V $\Vdash^+ \phi\uparrow f$. Then we
claim U$\Vdash$ U/id$\Vdash \phi \to \psi$, because for all g: W $\to$ U and all h: V $\to$ W, if
W$\Vdash$ V/h$\Vdash \phi\uparrow g \circ$ h, then V $\Vdash^+ \psi\uparrow g \circ$ h so V $\Vdash^+ \psi\uparrow g \circ$ h, that is W$\Vdash$ v/h$\Vdash \psi\uparrow g \circ$ h.
Conversely, if for all g: W $\to$ U and all h: Z $\to$ g*X, where f: X $\to$ U, if
W$\Vdash$ Z/g*f $\circ$ h $\Vdash*\psi\uparrow$f*g $\circ$ h, then X$\Vdash \phi \to \psi$, because for h: Z $\to$ X if Z$\Vdash \phi\uparrow$h then
U$\Vdash$ Z/f $\circ$ h $\Vdash * \phi\uparrow$h so U$\Vdash$ Z/f $\circ$ h $\Vdash*\psi\uparrow$h which gives Z$\Vdash \psi\uparrow$h, so U$\Vdash$ X/f $\Vdash* \phi \to \psi$.
The proof for $\forall^+$ is similar.   □

We view this thorem as asserting that in the topos $\mathbb{E}$ the naive notion of truth
given by the equivalence thesis is consonant with the theory of meaning given by
the notion of forcing over the site $\mathbb{C}$. Of course this may seem vacuous as it
appears that $\mathbb{C}$ is manufactured with this result in mind. However, in the case of
primary interest for this paper, the results of §4 allow us to regard $\mathbb{C}$ internally
as a full subcategory of Loc($\mathbb{E}$) equipped with the open cover topology. In fact,
if $\mathbb{C}$ is the category of separable locales, we may identify $\mathbb{C}$ as a category of
separable locales in $\mathbb{E}$. We shall deal with this, among other things, in a sequel
to this paper.

Given f: X $\to$ U we may view an element a of A(X) as a function:

$$U \Vdash a: X/f \to \underset{\sim}{A}.$$

This allows us to rephrase our theorem.

5.3 Corollary.   U$\Vdash$ X/f$\Vdash \phi(\bar{a})$ iff U$\Vdash \forall t \in$ X/f$\phi(\bar{a}(t))$.   □

We view this as a general form of the elimination theorem (cf. Troelstra [1977]

pp.33,79). The appropriate theory of continuous truth CT has an axiom for each clause in the definition of $X/f \Vdash \phi(\bar{a})$. For example, the clause for $\exists$ gives the axiom of local choice $\forall t \in X\ \exists y \phi(\bar{a}(t),y)$ iff $\exists$ open cover p: $Z \twoheadrightarrow X$ and continuous f: $Z \to Y$ such that $\forall z \phi(\bar{a}(p(z)),f(z))$. The translation $\tau\phi$ of a formula ϕ without free lawless variables is given by $\tau\phi \equiv_{\mathrm{def}} \Vdash \phi$.

CODA

A general notion of non-constructive object is given by interpretations in Grothendieck topoi. The process of iteration described in §5 shows how we may view (internal) truth in this interpretation as given by a non-standard theory of meaning. The clauses defining this give axioms for the corresponding theory of continuous truth CT and an "elimination" translation. By construction, CT $\vdash \phi \leftrightarrow \tau\phi$ and for formulae in the lawlike part of the language $\tau\phi \equiv \phi$. The proof theoretic content of the elimination;

$$\mathrm{CT} \vdash \phi \ \mathrm{iff} \ \mathrm{ID} \vdash \tau\phi,$$

requires formalisation of our treatment in an appropriate theory ID of inductive definitions. We do not undertake this here.

REFERENCES

Artin, M., Grothendieck, A., Verdier, J.L., Théorie des Topos et Cohomologie, Etale des Schémas (SGA4), (Lecture Notes in Math. 269, 270, Springer-Verlag, Berlin, 1972).

Beth, E.W., Semantical Considerations on Intuitionistic Logic, Indag. Math., 9(1947), p.572-7.

Boileau, André & Joyal, André, La logique des topos, J.S.L. 46(1981), p.6-16.

Brouwer, L.E.J., Cambridge Lectures on Intuitionism, D. van Dalen, ed. (Cambridge University Press, 1981).

Dummett, Michael, Elements of Intuitionism, (Oxford University Press, 1977).

Dummett, Michael, Truth and other enigmas, (Duckworth, London, 1978).

Fourman, Michael P., The logic of Topoi, in Handbook of Math. Logic (ed. Barwise, J.), (North-Holland, 1977), p.1053-90.

Fourman, Michael P., Notions of Choice Sequence, Proc. Brouwer Symposium, (ed. Troelstra, A. and van Dalen, D.), (North-Holland, 1982).

Fourman, Michael P. & Grayson, Robin J., Formal Spaces, Proc. Brouwer Symposium, (ed. Troelstra, A. and van Dalen, D.), (North-Holland, 1982).

Fourman, Michael P., T_1 spaces over topological sites, JPAA, (to appear), 1983.

Freyd, Peter, Aspects of Topoi, Bull. Austral. Math. Soc., 7(1972), p.1-76.

Isbell, John, Atomless parts of spaces, Math. Scand., 31(1972), p.5-32.

Johnstone, Peter T., Topos Theory, (Acad. Press, London, 1977).

Johnstone, Peter T., Stone spaces, (Acad. Press, London, 1982).

Joyal, André, & Tierney, Myles, An extension of the Galois theory of Grothendieck, preprint, 1982.

Kreisel, Georg, Lawless sequences of natural numbers. Comp. Math. 20(1968), p.222-48.

Makkai, Michael & Reyes, Gonzalo, First-Order Categorical Logic, (Lecture Notes in Math. 611, Springer-Verlag, 1977).

Moschovakis, Joan R., A topological interpretation of second-order intuitionistic arithmetic, Comp. Math., (3), 26(1973), p.261-75.

Scott, Dana S., Extending the topological interpretation to intuitionistic analysis, Comp. Math. 20(1968), 222-48.

Scott, Dana S., Identity and Existence in Intuitionistic Logic, in Proc. Durham Symposium, (ed. Fourman et al.) (Lecture Notes in Math. 753, Springer-Verlag, 1978), p.660-96.

Troelstra, Anne S., Choice Sequences, (Oxford University Press, 1977).

Wraith, Gavin C., Lectures on elementary topoi, Model theory and topoi, (ed. Lawvere F.W. et al.), (Lecture Notes in Math. 445, Springer-Verlag, Berlin, 1975), p.114-206.

Wright, Crispin, Wittgenstein's Philosophy of Mathematics, (Duckworth, 1981).