Coherence for cartesian closed bicategories, via normalisation-by-evaluation

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Cartesian closed bicategories

Cartesian closed categories ‘up to isomorphism’.

Examples:

- Generalised species and cartesian distributors
  particularly for applications in higher category theory
  (Fiore, Gambino, Hyland, Winskel), (Fiore & Joyal)
- Categorical algebra (operads)
  (Gambino & Joyal)
- Game semantics (concurrent games)
  (Yamada & Abramsky, Winskel *et al.*, Paquet)

Coherence

Internal monoids

In a category with finite products:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

$\boxed{\text{Unit law}}$

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\cong"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\cong"] \\
& M &
\end{tikzcd}$$

$\boxed{\text{Assoc. law}}$

$$\begin{tikzcd}
(M \times M) \times M \arrow[r, "\cong"] \arrow[d, "m \times M"'] & M \times (M \times M) \arrow[r, "M \times m"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[rr, "m"'] & & M
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In **Set**: monoids
In **Cat**: strict monoidal categories

Unit law

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Internal pseudomonoids

In **Cat**:

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Unit 2-cells

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\simeq"'] & M \times M \arrow[d, "m"] \arrow[d, phantom, shift right=20pt, "\overset{\lambda}{\cong}"] \arrow[d, phantom, shift left=20pt, "\overset{\rho}{\cong}"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\simeq"] \\
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data

Internal pseudomonoids

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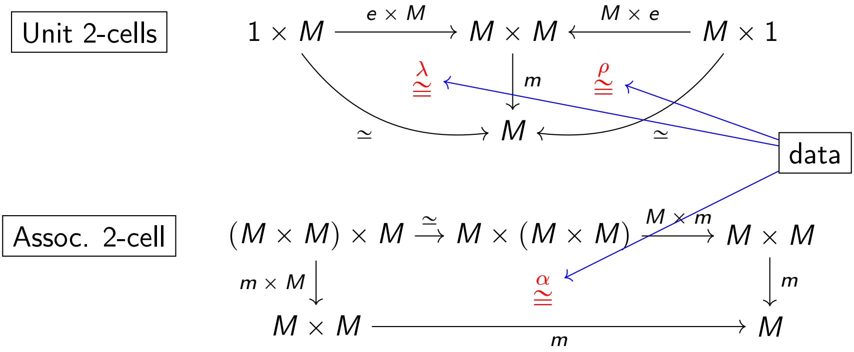

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+ triangle and pentagon laws ⇝ monoidal category

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$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

...likewise in any fp-bicategory

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+ triangle and pentagon laws

⇝ monoidal category

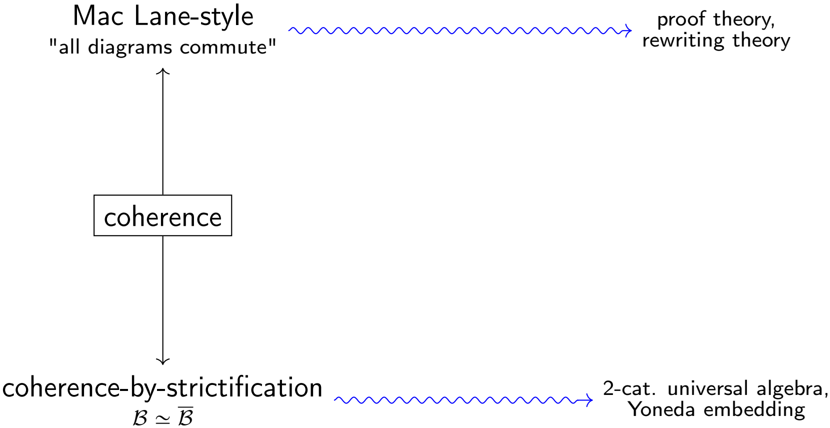
In a CCC every $[X \Rightarrow X]$ becomes a monoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

$\boxed{?}$ In a cc-bicategory every $[X \Rightarrow X]$ becomes a pseudomonoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

need to check
coherence laws
(*i.e.* triangle + pentagon)



Mac Lane-style

"all diagrams commute"

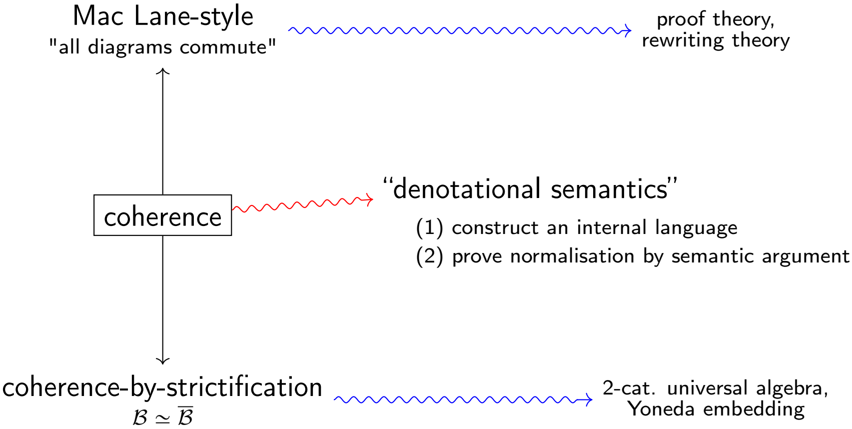
proof theory,
rewriting theory

coherence

coherence-by-strictification

$\mathcal{B} \simeq \overline{\mathcal{B}}$

2-cat. universal algebra,
Yoneda embedding



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“denotational semantics”

(1) construct an internal language

(2) prove normalisation by semantic argument

coherence-by-strictification

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2-cat. universal algebra,
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$\boxed{\text{coherence}} \rightsquigarrow$ “denotational semantics”

(1) construct an internal language
(2) prove normalisation by semantic argument

- builds on categorical & type-theoretic intuition
- *once set up* about as hard as categorical proof

Cartesian closed bicategories

Composition by UMP ⇒ bicategory

In a category $\mathbb{C}$ with pullbacks:

1. objects: objects of $\mathbb{C}$,
2. 1-cells $A \rightsquigarrow B$: spans $(A \leftarrow X \rightarrow B)$,
3. 2-cells: commutative squares

$$\begin{tikzcd}
& X \arrow[dd, dashed, "h"] \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & B \\
& Y \arrow[ur] &
\end{tikzcd}$$

Composition defined by pullback:

$$\begin{tikzcd}
& & X \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
& X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
A & & B & & C
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Composition defined by pullback: $\rightsquigarrow$ associative up to iso

$$\begin{tikzcd} & & X \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start, rotate=-45] & & \\ & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

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  2-cells $$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & Y \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "f'"{name=M, description}] \arrow[r, bend right=60, "f''"'{name=D}] & Y \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\alpha'"] \end{tikzcd}$$

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- Identities $\mathrm{Id}_X : X \to X$ and composition

$$\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \xrightarrow{\circ_{X,Y,Z}} \mathcal{B}(X,Z)$$

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$$\mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \xrightarrow{\circ_{X,Y,Z}} \mathcal{B}(X, Z)$$

- Invertible 2-cells

$$(h \circ g) \circ f \xRightarrow{\mathbf{a}_{h,g,f}} h \circ (g \circ f)$$

$$\mathrm{Id}_X \circ f \xRightarrow{\mathbf{l}_f} f$$

$$g \circ \mathrm{Id}_X \xRightarrow{\mathbf{r}_g} g$$

subject to a triangle law and pentagon law.

Cartesian closed bicategories

Bicategories $\mathcal{B}$ equipped with

Cartesian closed bicategories

Bicategories $\mathcal{B}$ equipped with families of equivalences

$$\mathcal{B}\big(X, \textstyle\prod_{i=1}^{n} A_i\big) \quad \simeq \quad \textstyle\prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

$$\mathcal{B}(X, A \Rightarrow B) \quad \simeq \quad \mathcal{B}(X \times A, B)$$

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

Cartesian closed bicategories

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$$\begin{tikzcd}
\mathcal{B}\left(X, \prod_{i=1}^{n} A_i\right) \arrow[r, bend left, "(\pi_{\mathbf{1}} \circ -, \dots, \pi_n \circ -)"] \arrow[r, phantom, "\perp\simeq"] & \prod_{i=1}^{n} \mathcal{B}(X, A_i) \arrow[l, bend left, "\langle -, \dots, = \rangle"]
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(tupling)

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$$\mathrm{eval}_{A,B} \circ (\lambda f \times A) \overset{\cong}{\Longrightarrow} f \qquad g \overset{\cong}{\Longrightarrow} \lambda(\mathrm{eval}_{A,B} \circ (g \times A))$$

The internal language for cc-bicategories, $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$

Judgements *c.f.* Hilken, Hirschowitz

Terms	$\Gamma \vdash t : A$	(1-cells)
Rewrites	$\Gamma \vdash \tau : t \Rightarrow t' : A$	(2-cells)
Equations	$\Gamma \vdash \tau \equiv \tau' : t \Rightarrow t' : A$	

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Features

- Weak composition enforced by *explicit substitution*

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash t : B \qquad (\Delta \vdash u_i : A_i)_{i=1.,n}}{\Delta \vdash t\left\{x_i \mapsto u_i\right\} : B}$$

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash \tau : t \Rightarrow t' : B \qquad (\Delta \vdash \sigma_i : u_i \Rightarrow u_i' : A_i)_{i=1,\dots,n}}{\Delta \vdash \tau\left\{x_i \mapsto \sigma_i\right\} : t\left\{x_i \mapsto u_i\right\} \Rightarrow t'\left\{x_i \mapsto u_i'\right\} : B}$$

$\rightsquigarrow$ binds the variables $x_1, \dots, x_n$

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- Free property for syntactic model

$$\begin{tikzcd}[column sep=huge] \llbracket \Gamma \rrbracket \arrow[r, bend left, "{\llbracket \Gamma \vdash t : A \rrbracket}"] \arrow[r, bend right, "{\llbracket \Gamma \vdash t' : A \rrbracket}"'] \arrow[r, phantom, "{\Downarrow \llbracket \Gamma \vdash \tau : t \Rightarrow t' : A \rrbracket}"] & \llbracket A \rrbracket \end{tikzcd}$$

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$\rightsquigarrow$ coherence of cc-bicategories = coherence property of 2-cells

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For any t and t' in $\Lambda_{\mathrm{ps}}^{\times,\to}$, there exists at most one rewrite τ such that $\Gamma \vdash \tau : t \Rightarrow t' : A$. □

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Corollary

For any $f, f' : X \rightarrow Y$ *in the free cc-bicategory on a graph, there exists at most one 2-cell* $\tau : f \Rightarrow f'$. □

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$\rightsquigarrow$ can use STLC for constructions in cc-bicategories!

1. prove result in STLC
2. $\beta\eta$-equalities $\rightsquigarrow$ 2-cells
3. coherence guaranteed

Normalisation-by-evaluation

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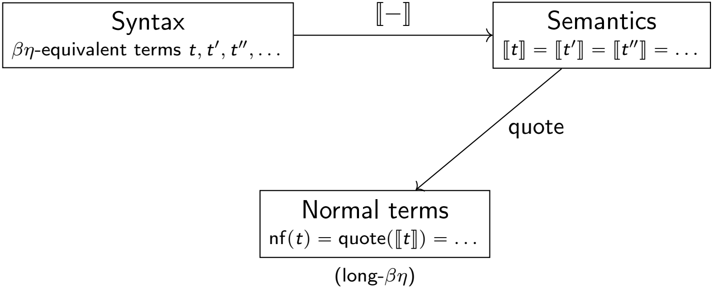
Aim: find canonical representatives within each $\beta\eta$-equivalence class

Strategy:



Syntax

$\beta\eta$-equivalent terms $t, t', t'', \dots$

$[\![-]\!]$

Semantics

$[\![t]\!] = [\![t']\!] = [\![t'']\!] = \dots$

quote

Normal terms

$\mathsf{nf}(t) = \mathsf{quote}([\![t]\!]) = \dots$

(long-$\beta\eta$)

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Strategy:

$$\begin{tikzcd}
\begin{array}{c}\text{Syntax}\\ \beta\eta\text{-equivalent terms } t, t', t'', \dots\end{array} \arrow[rr, "{[\![-]\!]}"] \arrow[dr, dashed, "\mathsf{nf}(-)"'] & & \begin{array}{c}\text{Semantics}\\ [\![t]\!] = [\![t']\!] = [\![t'']\!] = \dots\end{array} \arrow[dl, "\mathsf{quote}"] \\
& \begin{array}{c}\text{Normal terms}\\ \mathsf{nf}(t) = \mathsf{quote}([\![t]\!]) = \dots\end{array} &
\end{tikzcd}$$

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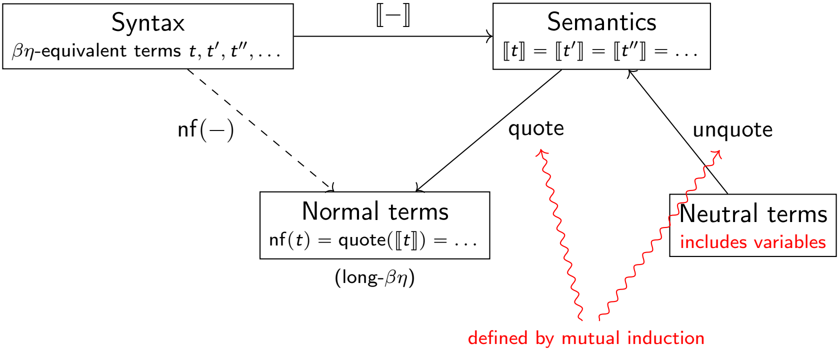
Aim: find canonical representatives within each $\beta\eta$-equivalence class

Strategy:



Syntax

$\beta\eta$-equivalent terms $t, t', t'', \ldots$

$[\![-]\!]$

Semantics

$[\![t]\!] = [\![t']\!] = [\![t'']\!] = \ldots$

nf$(-)$

quote

unquote

Normal terms

nf(t) = quote$([\![t]\!]) = \ldots$

(long-$\beta\eta$)

Neutral terms

includes variables

defined by mutual induction

Normalisation-by-evaluation

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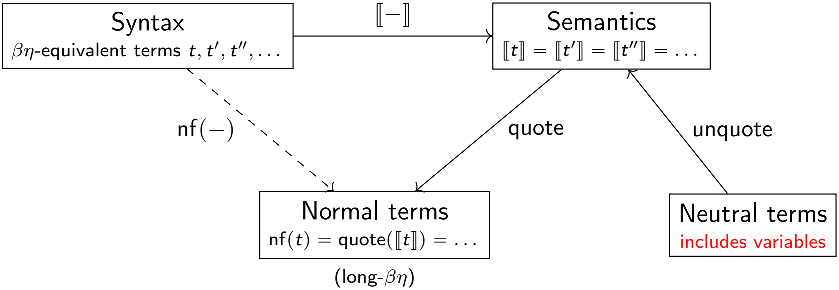

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For $\vdash \lambda x.t : A \Rightarrow B$, get $[\![t]\!] : [\![A]\!] \to [\![B]\!]$.

Normalisation-by-evaluation

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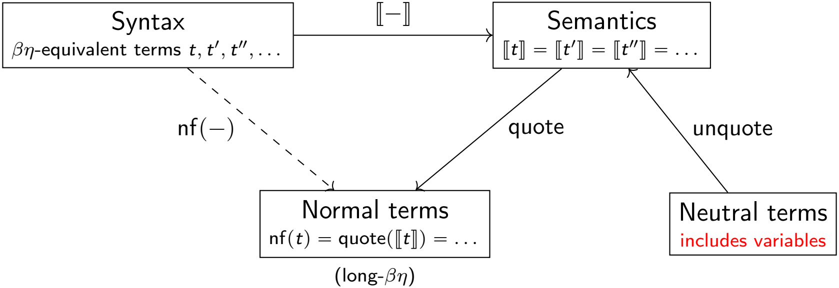

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$$\text{unquote}(x) : [\![A]\!]$$

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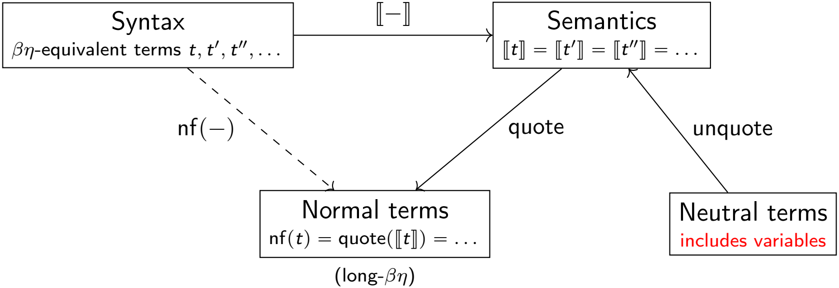

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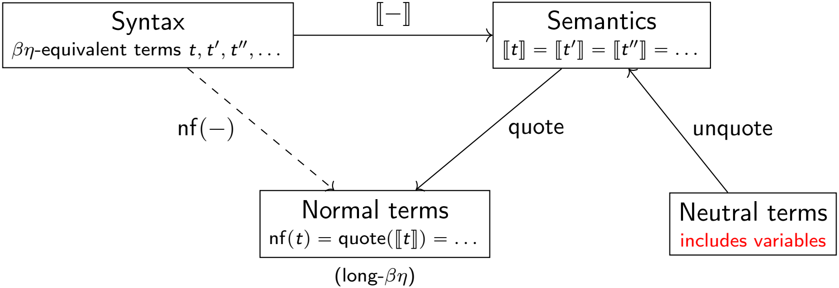

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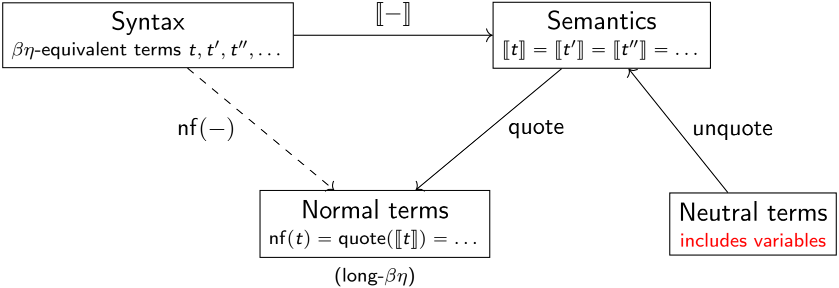

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For $\vdash \lambda x.t : A \Rightarrow B$, get $[\![t]\!] : [\![A]\!] \to [\![B]\!]$. For fresh x:

$$\mathsf{nf}(\lambda x.t) := \lambda x.\mathsf{quote}([\![t]\!](\mathsf{unquote}(x))) : A \Rightarrow B$$

Normalisation-by-evaluation, categorically

Normalisation-by-evaluation, categorically (Fiore)

Syntax as indexed presheaves over a *category of contexts* Con:

$$\left.\begin{array}{r}\mathsf{neuts}_A : \Gamma \mapsto \{\text{neutral terms } t \text{ such that } \Gamma \vdash t : A\} \\ \mathsf{norms}_A : \Gamma \mapsto \{\text{normal terms } t \text{ such that } \Gamma \vdash t : A\}\end{array}\right\} : \mathrm{Con} \to \mathbf{Set}$$

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syntax glued to semantics by natural transformations

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syntax glued to semantics by natural transformations

$$(\Gamma \vdash t : A) \mapsto s[\![\Gamma \vdash t : A]\!]$$

s any interpretation of base types;

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syntax glued to semantics by natural transformations

$$\begin{gathered}(\Gamma \vdash t : A) \mapsto s[\![\Gamma \vdash t : A]\!] \\ s[\![-]\!] : \mathsf{neuts}_A \Rightarrow \mathbb{C}(s[\![-]\!], s[\![A]\!]) \\ s[\![-]\!] : \mathsf{norms}_A \Rightarrow \mathbb{C}(s[\![-]\!], s[\![A]\!])\end{gathered}$$

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Strategy:

1. define a glueing category $\mathbb{G}$

Normalisation-by-evaluation, categorically (Fiore)

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2. pick an interpretation $\mathrm{e}[\![-]\!]$ in $\mathbb{G}$

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Strategy:

1. define a glueing category $\mathbb{G}$
2. pick an interpretation $\mathrm{e}[\![-]\!]$ in $\mathbb{G}$
3. define quote and unquote as maps in this category

The glued category $\mathbb{G}(\mathbb{C}, s)$

Objects

$\mathbb{C}$ ccc,
$s : \text{BaseTypes} \to \mathrm{Ob}\mathbb{C}$

nat. trans

functor
$\mathrm{Con} \to$ **Set**

$$\boxed{P \overset{p}{\Rightarrow} \mathbb{C}(s[\![-]\!], X)}$$

$\in \mathbb{C}$

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Morphisms

nat. trans $P \Rightarrow P'$

$$\begin{tikzcd}
P \arrow[r, "\kappa"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\
\mathbb{C}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] & \mathbb{C}(s[\![-]\!], X')
\end{tikzcd}$$

arrow $X \to X'$

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$\mathbb{C}$ ccc,
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"syntax"

"interpretation"

"semantics"

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The category $\mathbb{G}(\mathbb{C}, s)$

Objects

$\mathbb{C}$ ccc, s : BaseTypes $\to \mathrm{Ob}\mathbb{C}$

$$\begin{tikzcd}
& \text{nat. trans} \arrow[d, squiggly] & \\
\begin{array}{c}\text{functor}\\ \mathrm{Con} \to \textbf{Set}\end{array} \arrow[r, squiggly] & \boxed{P \overset{p}{\Rightarrow} \mathbb{C}(s[\![-]\!], X)} & \in \mathbb{C} \arrow[l, squiggly]
\end{tikzcd}$$

Morphisms

$$\begin{tikzcd}
\text{"syntax"} \arrow[r, squiggly, red] & P \arrow[r, "\kappa"] \arrow[dd, "p"'] & P' \arrow[dd, "p'"] \\
\text{"interpretation"} \arrow[r, squiggly, red] & {} & \\
\text{"semantics"} \arrow[r, squiggly, red] & \mathbb{C}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] & \mathbb{C}(s[\![-]\!], X')
\end{tikzcd}$$

The category $\mathbb{G}(\mathbb{C}, s)$

Objects

$\mathbb{C}$ ccc, $s : \mathrm{BaseTypes} \to \mathrm{Ob}\mathbb{C}$

$$\begin{tikzcd}
 & \text{nat. trans} \arrow[d, squiggly] & \\
\text{functor } \mathrm{Con} \to \mathbf{Set} \arrow[r, squiggly] & \boxed{P \overset{p}{\Rightarrow} \mathbb{C}(s[\![-]\!], X)} & \in \mathbb{C} \arrow[l, squiggly]
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Morphisms

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\text{"interpretation"} \arrow[r, squiggly] & & \\
\text{"semantics"} \arrow[r, squiggly] & \mathbb{C}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] & \mathbb{C}(s[\![-]\!], X')
\end{tikzcd}$$

$\rightsquigarrow$ Cartesian closed, with a strict CCC-functor $\mathbb{G}(\mathbb{C}, s) \to \mathbb{C}$

The category $\mathbb{G}(\mathbb{C}, s)$

Objects

$\mathbb{C}$ ccc, $s : \text{BaseTypes} \to \text{Ob}\mathbb{C}$

functor
$\text{Con} \to \textbf{Set}$

nat. trans

$$\boxed{P \overset{p}{\Rightarrow} \mathbb{C}(s[\![-]\!], X)} \quad \in \mathbb{C}$$

Morphisms

$$\begin{tikzcd}
\text{"syntax"} \arrow[r, squiggly, dash] & P \arrow[rr, "\kappa"] \arrow[dd, "p"'] & & P' \arrow[dd, "p'"] \\
\text{"interpretation"} \arrow[r, squiggly, dash] & {} & & \\
\text{"semantics"} \arrow[r, squiggly, dash] & \mathbb{C}(s[\![-]\!], X) \arrow[rr, "k \circ (-)"'] & & \mathbb{C}(s[\![-]\!], X')
\end{tikzcd}$$

$\rightsquigarrow$ Cartesian closed, with a strict CCC-functor $\mathbb{G}(\mathbb{C}, s) \to \mathbb{C}$

$\rightsquigarrow$ for every type A,

$$\left.\begin{array}{l}
\text{neuts}_A \xRightarrow{s[\![-]\!]} \mathbb{C}(s[\![-]\!], s[\![A]\!]) \\
\text{norms}_A \xRightarrow{s[\![-]\!]} \mathbb{C}(s[\![-]\!], s[\![A]\!])
\end{array}\right\} \in \mathbb{G}(\mathbb{C}, s)$$

Interpretation in the glued category $\mathbb{G}(\mathbb{C}, s)$

$\mathbb{C}$ any ccc
$s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathbb{C}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathbb{G}(\mathbb{C}, s))$$
$$\beta \mapsto \left(\mathsf{neuts}_\beta \xrightarrow{s[\![-]\!]} \mathbb{C}(s[\![-]\!], s[\![\beta]\!])\right)$$

Interpretation in the glued category $\mathbb{G}(\mathbb{C}, s)$

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$$
\begin{aligned}
\mathrm{e} : \text{BaseTypes} &\to \mathrm{Ob}(\mathbb{G}(\mathbb{C}, s)) \\
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\end{aligned}
$$

$$
\mathrm{e}[\![A]\!] := \left(\bar{\mathrm{e}}[\![A]\!] \overset{\nu_A}{\Longrightarrow} \mathbb{C}(s[\![-]\!], s[\![A]\!])\right)
$$

by strict preservation

Interpretation in the glued category $\mathbb{G}(\mathbb{C}, s)$

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$$\begin{tikzcd} \bar{\mathrm{e}}[\![A]\!] \arrow[d, "\nu_A"] \\ \mathbb{C}\big(s[\![-]\!], s[\![A]\!])\big) \end{tikzcd}$$

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$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathbb{C}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
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\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{neuts}_A \arrow[r, "\mathsf{quote}_A"] \arrow[d, "s\llbracket - \rrbracket"'] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathbb{C}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big) \arrow[r, equal, "\text{preserves } \beta\eta"'] & \mathbb{C}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

Interpretation in the glued category $\mathbb{G}(\mathbb{C}, s)$

$\mathbb{C}$ any ccc
s : BaseTypes $\to \mathrm{Ob}\mathbb{C}$

$$
\begin{aligned}
\mathrm{e} : \mathrm{BaseTypes} &\to \mathrm{Ob}(\mathbb{G}(\mathbb{C}, s)) \\
\beta &\mapsto \left(\mathrm{neuts}_\beta \xRightarrow{s\llbracket - \rrbracket} \mathbb{C}(s\llbracket - \rrbracket, s\llbracket \beta \rrbracket)\right)
\end{aligned}
$$

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\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{neuts}_A \arrow[r, "\mathrm{quote}_A"] \arrow[d, "s\llbracket - \rrbracket"'] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathbb{C}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right) \arrow[r, equal, "\text{preserves } \beta\eta"'] & \mathbb{C}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathrm{unquote}_A"] \arrow[d, "\nu_A"'] & \mathrm{norms}_A \arrow[d, "s\llbracket - \rrbracket"] \\
\mathbb{C}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right) \arrow[r, equal, "\text{preserves } \beta\eta"'] & \mathbb{C}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right)
\end{tikzcd}$$

Normalising $(\Gamma \vdash t : A)$

$$\begin{aligned} \mathsf{neuts}_\Gamma &:= \textstyle\prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i} \\ \mathsf{unquote}_\Gamma &:= \textstyle\prod_{(x_i : A_i) \in \Gamma} \mathsf{unquote}_{A_i} \end{aligned}$$

$$\begin{tikzcd}
\mathsf{neuts}_\Gamma \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[d, "\mathsf{s}\llbracket - \rrbracket"'] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A \arrow[d, "\mathsf{s}\llbracket - \rrbracket"] \\
\mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
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Normalising ($\Gamma \vdash t : A$)

$$\mathsf{neuts}_\Gamma := \prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

$$\mathsf{unquote}_\Gamma := \prod_{(x_i : A_i) \in \Gamma} \mathsf{unquote}_{A_i}$$

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\mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

$\rightsquigarrow$ recall $\mathsf{nf}(\lambda x.t) := \lambda x.\mathsf{quote}(\llbracket t \rrbracket(\mathsf{unquote}(x)))$

Normalising $(\Gamma \vdash t : A)$

$$\mathsf{neuts}_\Gamma := \textstyle\prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

$$\mathsf{unquote}_\Gamma := \textstyle\prod_{(x_i : A_i) \in \Gamma} \mathsf{unquote}_{A_i}$$

$$\begin{tikzcd}
\mathsf{neuts}_\Gamma \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[d, "\mathsf{s}\llbracket - \rrbracket"'] \arrow[rrr, bend left, "\mathsf{norm}_t"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A \arrow[d, "\mathsf{s}\llbracket - \rrbracket"] \\
\mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathbb{C}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

$\rightsquigarrow$ recall $\mathsf{nf}(\lambda x.t) := \lambda x.\mathsf{quote}(\llbracket t \rrbracket(\mathsf{unquote}(x)))$

Normalising $(\Gamma \vdash t : A)$

$$\mathsf{neuts}_\Gamma := \prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

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$$\begin{tikzcd}
\mathsf{neuts}_\Gamma(\Gamma) \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[rrr, bend left, "\mathsf{norm}^\Gamma_t"] \arrow[dr, bend right, "\mathsf{s}[\![-]\!]"'] & \overline{\mathrm{e}}[\![\Gamma]\!](\Gamma) \arrow[r, "\overline{\mathrm{e}}{[\![\Gamma \vdash t : A]\!]}"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}[\![A]\!](\Gamma) \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A(\Gamma) \arrow[dl, bend left, "s[\![-]\!]"] \\
 & \mathbb{C}(s[\![\Gamma]\!], s[\![\Gamma]\!]) \arrow[r, "s{[\![\Gamma \vdash t : A]\!]} \circ (-)"'] & \mathbb{C}(s[\![\Gamma]\!], s[\![A]\!]) &
\end{tikzcd}$$

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$$\mathsf{neuts}_\Gamma := \prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

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$$\begin{tikzcd}
\mathsf{neuts}_\Gamma(\Gamma) \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[rrr, bend left, "\mathsf{norm}_t^\Gamma"] \arrow[dr, bend right, "\mathsf{s}\llbracket - \rrbracket"'] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket(\Gamma) \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}\llbracket A \rrbracket(\Gamma) \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A(\Gamma) \arrow[dl, bend left, "s\llbracket - \rrbracket"] \\
 & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket A \rrbracket) &
\end{tikzcd}$$

Define $\mathsf{nf}(t) := \mathsf{norm}_t^\Gamma((\Gamma \vdash x_i : A_i)_{i=1,\ldots,n})$.

Normalising $(\Gamma \vdash t : A)$

$$\mathsf{neuts}_\Gamma := \textstyle\prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

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$(x_i : A_i)_i$

$$\begin{tikzcd}
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 & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket A \rrbracket) & \\
 & \mathrm{id}_{s\llbracket \Gamma \rrbracket} & &
\end{tikzcd}$$

Define $\mathsf{nf}(t) := \mathsf{norm}^\Gamma_t((\Gamma \vdash x_i : A_i)_{i=1,\ldots,n})$.

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 & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket A \rrbracket) & \\
 & & & s\llbracket \Gamma \vdash t : A \rrbracket
\end{tikzcd}$$

Define $\mathsf{nf}(t) := \mathsf{norm}^\Gamma_t((\Gamma \vdash x_i : A_i)_{i=1,\ldots,n})$.

Normalising $(\Gamma \vdash t : A)$

$$\mathsf{neuts}_\Gamma := \prod_{(x_i : A_i) \in \Gamma} \mathsf{neuts}_{A_i}$$

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$$\begin{tikzcd}
(x_i : A_i)_i & \mathsf{neuts}_\Gamma(\Gamma) \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[rrrr, bend left, "\mathsf{norm}^\Gamma_t"] \arrow[dr, bend right, "\mathsf{s}\llbracket - \rrbracket"'] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket(\Gamma) \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overline{\mathrm{e}}\llbracket A \rrbracket(\Gamma) \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A(\Gamma) \arrow[dl, bend left, "s\llbracket - \rrbracket"] \\
& & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathbb{C}(s\llbracket \Gamma \rrbracket, s\llbracket A \rrbracket) &
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& \mathbb{C}(s[\![\Gamma]\!], s[\![\Gamma]\!]) \arrow[r, "{s[\![\Gamma \vdash t : A]\!] \circ (-)}"'] & \mathbb{C}(s[\![\Gamma]\!], s[\![A]\!]) & &
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$$\begin{tikzcd}
\mathsf{neuts}_\Gamma(\Gamma) \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[rrr, bend left, "\mathsf{norm}^\Gamma_t"] \arrow[dr, bend right, swap, "\mathsf{s}[\![-]\!]"] & \overline{\mathrm{e}}[\![\Gamma]\!](\Gamma) \arrow[r, "{\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]}"] \arrow[d, swap, "\nu_\Gamma"] & \overline{\mathrm{e}}[\![A]\!](\Gamma) \arrow[r, "\mathsf{quote}_A"] \arrow[d, "\nu_A"] & \mathsf{norms}_A(\Gamma) \arrow[dl, bend left, "\mathsf{s}[\![-]\!]"] \\
 & \mathbb{C}(s[\![\Gamma]\!], s[\![\Gamma]\!]) \arrow[r, swap, "{s[\![\Gamma \vdash t : A]\!] \circ (-)}"] & \mathbb{C}(s[\![\Gamma]\!], s[\![A]\!]) &
\end{tikzcd}$$

$$s[\![\Gamma \vdash t : A]\!]$$

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\end{tikzcd}$$

Define $\mathsf{nf}(t) := \mathsf{norm}^\Gamma_t((\Gamma \vdash x_i : A_i)_{i=1,\dots,n})$.

Since $s\llbracket \Gamma \vdash \mathsf{nf}(t) : A \rrbracket = s\llbracket \Gamma \vdash t : A \rrbracket$ in every model, $\mathsf{nf}(t) =_{\beta\eta} t$.

Normalisation-by-evaluation for $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$

Strategy

1. define a glueing bicategory $\mathcal{G}$
2. pick an interpretation $\mathrm{e}[\![-]\!]$ in $\mathcal{G}$
3. define quote and unquote as maps in this bicategory

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,... bicategorically)

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,... bicategorically)

$\rightsquigarrow$ use embedding of STLC into $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$

$$\left.\begin{aligned} \mathsf{neuts}_A^{\mathrm{ps}} &:= \mathsf{neuts}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\rightarrow} \\ \mathsf{norms}_A^{\mathrm{ps}} &:= \mathsf{norms}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\rightarrow} \end{aligned}\right\} : \mathrm{Con} \rightarrow \mathbf{Set}$$

$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,... bicategorically)

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,... bicategorically)

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$

$$s[\![t]\!] =\!=\!= s[\![\mathsf{nf}(t)]\!]$$

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,... bicategorically)

$\rightsquigarrow$ use embedding of STLC into $\Lambda_{\mathrm{ps}}^{\times,\to}$

$$\left.\begin{array}{r} \mathsf{neuts}_A^{\mathrm{ps}} := \mathsf{neuts}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\to} \\ \mathsf{norms}_A^{\mathrm{ps}} := \mathsf{norms}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\to} \end{array}\right\} : \mathrm{Con} \to \mathbf{Cat}$$

$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$

depends on t

$$s[\![t]\!] \xrightarrow{\cong} s[\![\mathsf{nf}(t)]\!]$$

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,… bicategorically)

$\rightsquigarrow$ use embedding of STLC into $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$

$$\underset{}{\overset{\text{depends on } t}{s[\![t]\!] \xRightarrow{\cong} s[\![\mathsf{nf}(t)]\!]}}$$

$$\underset{\text{depends on } t'}{s[\![t']\!] \xRightarrow[\cong]{} s[\![\mathsf{nf}(t')]\!]}$$

Strategy

- re-use universal properties

  (quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,… bicategorically)

- use embedding of STLC into $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$

$$\left.\begin{array}{r}\mathsf{neuts}_A^{\mathrm{ps}} := \mathsf{neuts}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\rightarrow} \\ \mathsf{norms}_A^{\mathrm{ps}} := \mathsf{norms}_A^{\mathrm{STLC}} \text{ inside } \Lambda_{\mathrm{ps}}^{\times,\rightarrow}\end{array}\right\} : \mathrm{Con} \rightarrow \mathbf{Cat}$$

$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

-

depends on t

$$\begin{tikzcd}
s[\![t]\!] \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "s[\![\tau]\!]"'] & s[\![\mathsf{nf}(t)]\!] \arrow[d, Rightarrow] \\
s[\![t']\!] \arrow[r, Rightarrow, "\cong"'] & s[\![\mathsf{nf}(t')]\!]
\end{tikzcd}$$

depends on t'

Strategy

$\rightsquigarrow$ re-use universal properties

(quote, unquote, $\mathbb{G}(\mathbb{C}, s)$,. . . bicategorically)

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$

depends on t

$$\begin{tikzcd} s[\![t]\!] \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "s[\![\tau]\!]"'] & s[\![\mathsf{nf}(t)]\!] \arrow[d, equal] \\ s[\![t']\!] \arrow[r, Rightarrow, "\cong"'] & s[\![\mathsf{nf}(t')]\!] \end{tikzcd}$$

depends on t'

Strategy

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$

depends on t

$$\begin{tikzcd} t \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\tau"'] & \mathsf{nf}(t) \arrow[d, equal] \\ t' \arrow[r, Rightarrow, "\cong"'] & \mathsf{nf}(t') \end{tikzcd}$$

depends on t'

Strategy

$\rightsquigarrow$ re-use universal properties

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$\mathsf{neuts}_A^{\mathrm{ps}}(\Gamma), \mathsf{norms}_A^{\mathrm{ps}}(\Gamma)$ sets of terms as discrete categories

$\rightsquigarrow$ If τ exists, it's unique

depends on t

$$\begin{tikzcd}
t \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "\tau"'] & \mathsf{nf}(t) \arrow[d, equal] \\
t' \arrow[r, Rightarrow, "\cong"'] & \mathsf{nf}(t')
\end{tikzcd}$$

depends on t'

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat
s any interpretation of base types

Want: for every type A,

$$\left.\begin{aligned} \mathsf{neuts}^{\mathrm{ps}}_{A} &\xRightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![A]\!]) \\ \mathsf{norms}^{\mathrm{ps}}_{A} &\xRightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![A]\!]) \end{aligned}\right\} \in \mathcal{G}(\mathcal{B}, s)$$

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

Objects

$\mathcal{B}$ cc-bicat,
s : BaseTypes $\to \mathrm{Ob}\mathcal{B}$

pseudo
nat. trans.

pseudofunctor
$\mathrm{Con} \to$ **Cat**

$$\boxed{P \xRightarrow{(p,\overline{p})} \mathcal{B}(s[\![-]\!], X)}$$

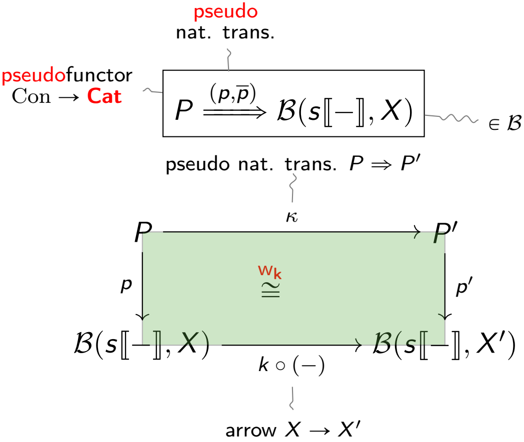
$\in \mathcal{B}$

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

Objects

$\mathcal{B}$ cc-bicat,
$s : \text{BaseTypes} \to \mathrm{Ob}\mathcal{B}$

1-cells



pseudo nat. trans.

pseudofunctor $\mathrm{Con} \to$ **Cat**

$$\boxed{P \xRightarrow{(p,\overline{p})} \mathcal{B}(s[\![-]\!], X)}$$

$\in \mathcal{B}$

pseudo nat. trans. $P \Rightarrow P'$

$$\begin{tikzcd}
P \arrow[r, "\kappa"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{w_k}{\cong}"] & P' \arrow[d, "p'"] \\
\mathcal{B}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] & \mathcal{B}(s[\![-]\!], X')
\end{tikzcd}$$

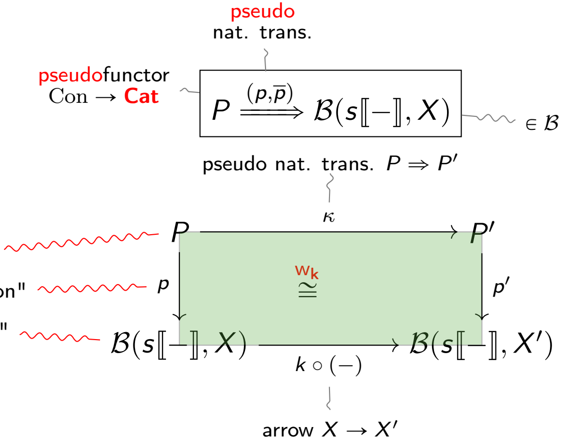
arrow $X \to X'$

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

Objects

$\mathcal{B}$ cc-bicat,

$s : \text{BaseTypes} \to \text{Ob}\mathcal{B}$



pseudo nat. trans.

pseudofunctor $\text{Con} \to \textbf{Cat}$

$$\boxed{P \xRightarrow{(p,\overline{p})} \mathcal{B}(s[\![-]\!], X)}$$

$\in \mathcal{B}$

1-cells

pseudo nat. trans. $P \Rightarrow P'$

"syntax"

"interpretation"

"semantics"

$$\begin{tikzcd} P \arrow[r, "\kappa"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\ \mathcal{B}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] \arrow[ur, phantom, "\overset{w_k}{\cong}"] & \mathcal{B}(s[\![-]\!], X') \end{tikzcd}$$

arrow $X \to X'$

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

Objects

$\mathcal{B}$ cc-bicat,
s : BaseTypes → Ob$\mathcal{B}$

pseudo
nat. trans.

pseudofunctor
Con → **Cat**

$$\boxed{P \xRightarrow{(p,\overline{p})} \mathcal{B}(s[\![-]\!], X)}$$

$\in \mathcal{B}$

1-cells

"syntax"
"interpretation"
"semantics"

$$\begin{tikzcd}
P \arrow[r, "\kappa"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\mathsf{w_k}}{\cong}"] & P' \arrow[d, "p'"] \\
\mathcal{B}(s[\![-]\!], X) \arrow[r, "k \circ (-)"'] & \mathcal{B}(s[\![-]\!], X')
\end{tikzcd}$$

2-cells

$$\begin{tikzcd}
P \arrow[r, bend left, "\kappa", ""{name=U, below}] \arrow[r, bend right, "\kappa'"', ""{name=D, above}] & P' \arrow[from=U, to=D, Rightarrow, "\Theta"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}(s[\![-]\!], X) \arrow[r, bend left, "k \circ (-)", ""{name=U, below}] \arrow[r, bend right, "k' \circ (-)"', ""{name=D, above}] & \mathcal{B}(s[\![-]\!], X') \arrow[from=U, to=D, Rightarrow, "\xi \circ (-)"]
\end{tikzcd}$$

The glued cc-bicategory $\mathcal{G}(\mathcal{B}, s)$

$$\begin{tikzcd}
P \arrow[rr, bend left, "\kappa"{name=U}] \arrow[rr, bend right, "\kappa'"'{name=D}] \arrow[d] & & P' \arrow[d] \\
\mathcal{B}(s\llbracket - \rrbracket, X) \arrow[rr, bend right, "k' \circ (-)"'] & \overset{\mathrm{w}_{k'}}{\cong} & \mathcal{B}(s\llbracket - \rrbracket, X')
\arrow[from=U, to=D, Rightarrow, "\Theta"]
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
P \arrow[rr, bend left, "\kappa"] \arrow[d] & \overset{\mathrm{w}_{k}}{\cong} & P' \arrow[d] \\
\mathcal{B}(s\llbracket - \rrbracket, X) \arrow[rr, bend left, "k \circ (-)"{name=U}] \arrow[rr, bend right, "k' \circ (-)"'{name=D}] & & \mathcal{B}(s\llbracket - \rrbracket, X')
\arrow[from=U, to=D, Rightarrow, "\xi \circ (-)"]
\end{tikzcd}$$

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat
s : BaseTypes $\to \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}^{\mathrm{ps}}_{\beta} \xrightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![\beta]\!])\right)$$

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat

s : BaseTypes $\rightarrow \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \rightarrow \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}^{\mathrm{ps}}_{\beta} \xRightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![\beta]\!])\right)$$

$$\mathrm{e}[\![A]\!] := \left(\bar{\mathrm{e}}[\![A]\!] \overset{\nu_A}{\Longrightarrow} \mathcal{B}(s[\![-]\!], s[\![A]\!])\right)$$

by strict preservation

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat
$s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}_{\beta}^{\mathrm{ps}} \xRightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![\beta]\!])\right)$$

$$\begin{tikzcd}
\bar{\mathrm{e}}[\![A]\!] \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s[\![-]\!], s[\![A]\!]\big)
\end{tikzcd}$$

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat

$s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}^{\mathrm{ps}}_{\beta} \xRightarrow{s[\![-]\!]} \mathcal{B}(s[\![-]\!], s[\![\beta]\!])\right)$$

$$\begin{tikzcd}
\overline{\mathrm{e}}[\![\Gamma]\!] \arrow[r, "\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]"] \arrow[d, "\nu_\Gamma"'] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}[\![A]\!] \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s[\![-]\!], s[\![\Gamma]\!]\big) \arrow[r, "s[\![\Gamma \vdash t : A]\!] \circ (-)"'] & \mathcal{B}\big(s[\![-]\!], s[\![A]\!]\big)
\end{tikzcd}$$

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat
$s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}^{\mathrm{ps}}_{\beta} \xRightarrow{s\llbracket - \rrbracket} \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \beta \rrbracket)\right)$$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[rr, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_{\Gamma}"'] & \overset{\mathsf{w}_t}{\cong} & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\left(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\right) \arrow[rr, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & & \mathcal{B}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{A} \arrow[rr, "\mathsf{quote}_A"] \arrow[d, "s\llbracket - \rrbracket"'] & \overset{\overline{q}_A}{\cong} & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\left(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\right) \arrow[rr, equal] & & \mathcal{B}\left(s\llbracket - \rrbracket, s\llbracket A \rrbracket\right)
\end{tikzcd}$$

preserves $\beta\eta$ mod $\cong$

Interpretation in $\mathcal{G}(\mathcal{B}, s)$

$\mathcal{B}$ any cc-bicat

s : BaseTypes $\to \mathrm{Ob}\mathcal{B}$

$$\mathrm{e} : \mathsf{BaseTypes} \to \mathrm{Ob}(\mathcal{G}(\mathcal{B}, s))$$

$$\beta \mapsto \left(\mathsf{neuts}^{\mathrm{ps}}_{\beta} \xRightarrow{s\llbracket - \rrbracket} \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \beta \rrbracket)\right)$$

$$\begin{tikzcd}
\bar{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\bar{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \bar{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{A} \arrow[r, "\mathsf{quote}_A"] \arrow[d, "s\llbracket - \rrbracket"'] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \bar{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

preserves $\beta\eta$ mod $\cong$

$$\begin{tikzcd}
\bar{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{unquote}_A"] \arrow[d, "\nu_A"'] \arrow[dr, phantom, "\overset{\overline{u}_A}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "s\llbracket - \rrbracket"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

preserves $\beta\eta$ mod $\cong$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[d, "\mathsf{s}\llbracket - \rrbracket"'] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_{A}"] \arrow[d] \arrow[dr, phantom, "\overset{\overline{q}_{A}}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "s\llbracket - \rrbracket"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[d, "\mathsf{s}[\![-]\!]"'] \arrow[rrr, bend left, "\mathsf{norm}_t"] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}[\![\Gamma]\!] \arrow[r, "\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]"] \arrow[d] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}[\![A]\!] \arrow[r, "\mathsf{quote}_A"] \arrow[d] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "s[\![-]\!]"] \\
\mathcal{B}(s[\![-]\!], s[\![\Gamma]\!]) \arrow[r, equal] & \mathcal{B}(s[\![-]\!], s[\![\Gamma]\!]) \arrow[r, "s[\![\Gamma \vdash t : A]\!] \circ (-)"'] & \mathcal{B}(s[\![-]\!], s[\![A]\!]) \arrow[r, equal] & \mathcal{B}(s[\![-]\!], s[\![A]\!])
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[d, "s\llbracket - \rrbracket"'] \arrow[rrr, bend left, "\mathsf{norm}_t"] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_A"] \arrow[d] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "s\llbracket - \rrbracket"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

Instantiate at Γ, pass in variables $\rightsquigarrow$ $\boxed{s\llbracket t \rrbracket \cong s\llbracket \mathsf{nf}(t) \rrbracket}$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[d, "\mathsf{s}\llbracket - \rrbracket"'] \arrow[rrr, bend left, "\mathsf{norm}_t"] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_A"] \arrow[d] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "s\llbracket - \rrbracket"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

Instantiate at Γ, pass in variables $\rightsquigarrow$ $\boxed{s\llbracket t \rrbracket \cong s\llbracket \mathsf{nf}(t) \rrbracket}$

$$s\llbracket t \rrbracket \xRightarrow{\cong} s\llbracket \mathsf{nf}(t) \rrbracket$$

$$s\llbracket t' \rrbracket \xRightarrow[\cong]{} s\llbracket \mathsf{nf}(t') \rrbracket$$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[d, "\mathsf{s}\llbracket - \rrbracket"'] \arrow[rrr, bend left, "\mathsf{norm}_t"] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d] \arrow[dr, phantom, "\overset{\mathsf{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_{A}"] \arrow[d] \arrow[dr, phantom, "\overset{\overline{q}_{A}}{\cong}"] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[d, "\mathsf{s}\llbracket - \rrbracket"] \\
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[r, equal] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

Instantiate at Γ, pass in variables ⇝ $\boxed{s\llbracket t \rrbracket \cong s\llbracket \mathsf{nf}(t) \rrbracket}$

$$\begin{tikzcd}
s\llbracket t \rrbracket \arrow[r, Rightarrow, "\cong"] \arrow[d, Rightarrow, "s\llbracket \tau \rrbracket"'] & s\llbracket \mathsf{nf}(t) \rrbracket \arrow[d, Rightarrow] \\
s\llbracket t' \rrbracket \arrow[r, Rightarrow, "\cong"'] & s\llbracket \mathsf{nf}(t') \rrbracket
\end{tikzcd}$$

Proving coherence

Interpreting $(\Gamma \vdash \tau : t \Rightarrow t' : A)$ in $\mathcal{G}(\mathcal{B}, s)$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[rr, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
& \overset{\mathsf{w}_t}{\cong} & \\
\mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[rr, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & & \mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

Interpreting $(\Gamma \vdash \tau : t \Rightarrow t' : A)$ in $\mathcal{G}(\mathcal{B}, s)$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[rr, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overset{\mathrm{w}_t}{\cong} & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[rr, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & & \mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[rr, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t' : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] & \overset{\mathrm{w}_{t'}}{\cong} & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[rr, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] & & \mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

Interpreting $(\Gamma \vdash \tau : t \Rightarrow t' : A)$ in $\mathcal{G}(\mathcal{B}, s)$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] \arrow[dr, phantom, "\overset{\mathrm{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[r, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"'] & \mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t' : A \rrbracket"] \arrow[d, "\nu_\Gamma"'] \arrow[dr, phantom, "\overset{\mathrm{w}_{t'}}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[d, "\nu_A"] \\
\mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket\big) \arrow[r, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] & \mathcal{B}\big(s\llbracket - \rrbracket, s\llbracket A \rrbracket\big)
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[r, bend right, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t' : A \rrbracket"'] \arrow[r, phantom, "\Downarrow\, \overline{\mathrm{e}}\llbracket \tau \rrbracket"] & \overline{\mathrm{e}}\llbracket A \rrbracket
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, bend left, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"] \arrow[r, bend right, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] \arrow[r, phantom, "\Downarrow\, s\llbracket \tau \rrbracket \circ (-)"] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

Cylinder condition for $(\Gamma \vdash \tau : t \Rightarrow t' : A)$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket\Gamma\rrbracket \arrow[rr, bend left, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t:A\rrbracket"] \arrow[rr, bend right, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t':A\rrbracket"'] \arrow[d] & \Downarrow \overline{\mathrm{e}}\llbracket\tau\rrbracket & \overline{\mathrm{e}}\llbracket A\rrbracket \arrow[d] \\
\mathcal{B}(s\llbracket-\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[rr, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] & \overset{\mathsf{w}_{t'}}{\cong} & \mathcal{B}(s\llbracket-\rrbracket, s\llbracket A\rrbracket)
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
\overline{\mathrm{e}}\llbracket\Gamma\rrbracket \arrow[rr, bend left, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t:A\rrbracket"] \arrow[d] & \overset{\mathsf{w}_{t}}{\cong} & \overline{\mathrm{e}}\llbracket A\rrbracket \arrow[d] \\
\mathcal{B}(s\llbracket-\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[rr, bend left, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"] \arrow[rr, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] & \Downarrow s\llbracket\tau\rrbracket\circ(-) & \mathcal{B}(s\llbracket-\rrbracket, s\llbracket A\rrbracket)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[dr, bend right, "s\llbracket - \rrbracket"'] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[d] \arrow[r, phantom, "\overset{\mathrm{w}_t}{\cong}"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_{A}"] \arrow[d] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[dl, bend left, "s\llbracket - \rrbracket"] \arrow[dl, phantom, "\overset{\overline{q}_{A}}{\cong}"] \\
 & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, bend left, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"{name=U}] \arrow[r, bend right, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'{name=D}] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) &
\arrow[Rightarrow, from=U, to=D, "s\llbracket \tau \rrbracket \circ (-)"]
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_{\Gamma}"] \arrow[dr, bend right, "s\llbracket - \rrbracket"'] \arrow[dr, phantom, "\overset{\overline{u}_{\Gamma}}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"{name=U}] \arrow[r, bend right, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t' : A \rrbracket"'{name=D}] \arrow[d] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[r, "\mathsf{quote}_{A}"] \arrow[d] & \mathsf{norms}^{\mathrm{ps}}_{A} \arrow[dl, bend left, "s\llbracket - \rrbracket"] \arrow[dl, phantom, "\overset{\overline{q}_{A}}{\cong}"] \\
 & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, bend right, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] \arrow[r, phantom, "\overset{\mathrm{w}_{t'}}{\cong}"] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) &
\arrow[Rightarrow, from=U, to=D, "\overline{\mathrm{e}}\llbracket \tau \rrbracket"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[rr, bend left, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"] \arrow[rr, bend right, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] \arrow[rr, phantom, "\Downarrow s\llbracket \tau \rrbracket \circ (-)"] & & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket)
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[dl, bend right, "s\llbracket - \rrbracket"'] \arrow[r, bend left, "s\llbracket \Gamma \vdash t : A \rrbracket \circ (-)"] \arrow[r, phantom, "\overset{\mathrm{w}_t^{-1}}{\cong}"] \arrow[dl, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) \arrow[dr, bend left, "s\llbracket - \rrbracket"] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \\
\mathrm{neuts}^{\mathrm{ps}}_{\Gamma} \arrow[r, "\mathsf{unquote}_\Gamma"] \arrow[dr, bend right, "s\llbracket - \rrbracket"'] \arrow[dr, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \overline{\mathrm{e}}\llbracket \Gamma \rrbracket \arrow[u] \arrow[d] \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t : A \rrbracket"] \arrow[r, bend right, "\overline{\mathrm{e}}\llbracket \Gamma \vdash t' : A \rrbracket"'] \arrow[r, phantom, "\Downarrow \overline{\mathrm{e}}\llbracket \tau \rrbracket"] & \overline{\mathrm{e}}\llbracket A \rrbracket \arrow[u] \arrow[d] \arrow[r, "\mathsf{quote}_A"] & \mathrm{norms}^{\mathrm{ps}}_{A} \arrow[dl, bend left, "s\llbracket - \rrbracket"] \arrow[dl, phantom, "\overset{\overline{q}_A}{\cong}"] \\
 & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket \Gamma \rrbracket) \arrow[r, bend right, "s\llbracket \Gamma \vdash t' : A \rrbracket \circ (-)"'] \arrow[r, phantom, "\overset{\mathrm{w}_{t'}}{\cong}"] & \mathcal{B}(s\llbracket - \rrbracket, s\llbracket A \rrbracket) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend left, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"{name=U}] \arrow[r, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'{name=D}] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket)
\arrow[Rightarrow, from=U, to=D, "s\llbracket\tau\rrbracket\circ(-)"]
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[dl, bend right, "s\llbracket-\rrbracket"'] \arrow[r, bend left, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"] \arrow[r, phantom, "\overset{\mathrm{w}_t^{-1}}{\cong}"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) \arrow[dr, bend left, "s\llbracket-\rrbracket"] & \\
\mathrm{neuts}^{\mathrm{ps}}_{\Gamma}(\Gamma) \arrow[r] \arrow[dr, bend right, "s\llbracket-\rrbracket"'] \arrow[ur, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] \arrow[dr, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \mathrm{e}\llbracket\Gamma\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t:A\rrbracket"{name=U}] \arrow[r, bend right, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t':A\rrbracket"'{name=D}] & \mathrm{e}\llbracket A\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r] & \mathrm{norms}^{\mathrm{ps}}_{A}(\Gamma) \arrow[dl, bend left, "s\llbracket-\rrbracket"] \arrow[ul, phantom, "\overset{\overline{q}_A}{\cong}"] \arrow[dl, phantom, "\overset{\overline{q}_A}{\cong}"] \\
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] \arrow[r, phantom, "\overset{\mathrm{w}_{t'}}{\cong}"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) &
\arrow[Rightarrow, from=U, to=D, "\overline{\mathrm{e}}\llbracket\tau\rrbracket"]
\end{tikzcd}$$

$\boxed{\text{In } \mathrm{Hom}(\mathrm{Con}, \mathbf{Cat})}$

$$\begin{tikzcd}
\mathsf{neuts}_{\Gamma}^{\mathrm{ps}} \arrow[r, "\mathsf{unquote}_{\Gamma}"] & \overline{\mathrm{e}}[\![\Gamma]\!] \arrow[r, bend left, "\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]"] \arrow[r, bend right, "\overline{\mathrm{e}}[\![\Gamma \vdash t' : A]\!]"'] \arrow[r, phantom, "\Downarrow \overline{\mathrm{e}}[\![\tau]\!]"] & \overline{\mathrm{e}}[\![A]\!] \arrow[r, "\mathsf{quote}_{A}"] & \mathsf{norms}_{A}^{\mathrm{ps}}
\end{tikzcd}$$

In **Cat**

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma}(\Delta) \arrow[r, "\mathsf{unquote}_{\Gamma}"] & \overline{\mathrm{e}}[\![\Gamma]\!](\Delta) \arrow[r, bend left, "\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]"] \arrow[r, bend right, "\overline{\mathrm{e}}[\![\Gamma \vdash t' : A]\!]"'] \arrow[r, phantom, "\Downarrow \overline{\mathrm{e}}[\![\tau]\!]"] & \overline{\mathrm{e}}[\![A]\!](\Delta) \arrow[r, "\mathsf{quote}_{A}"] & \mathsf{norms}^{\mathrm{ps}}_{A}(\Delta) \\
 & & & \text{a set!}
\end{tikzcd}$$

$\boxed{\text{In } \mathbf{Cat}}$

$$\begin{tikzcd}
\mathsf{neuts}^{\mathrm{ps}}_{\Gamma}(\Delta) \arrow[r, "\mathsf{unquote}_{\Gamma}"] & \overline{\mathrm{e}}[\![\Gamma]\!](\Delta) \arrow[rr, bend left, "{\overline{\mathrm{e}}[\![\Gamma \vdash t : A]\!]}"] \arrow[rr, bend right, "{\overline{\mathrm{e}}[\![\Gamma \vdash t' : A]\!]}"'] & \Downarrow \overline{\mathrm{e}}[\![\tau]\!] & \overline{\mathrm{e}}[\![A]\!](\Delta) \arrow[r, "\mathsf{quote}_{A}"] & \mathsf{norms}^{\mathrm{ps}}_{A}(\Delta) \\
 & & & & \text{a set!}
\end{tikzcd}$$

Composite can only be the identity

$$\begin{tikzcd}[column sep=huge]
\mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend left=40, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)", ""{name=U, below}] \arrow[r, bend right=40, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"', ""{name=D, above}] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket)
\arrow[Rightarrow, from=U, to=D, "s\llbracket\tau\rrbracket\circ(-)"]
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}[column sep=large, row sep=large]
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend left=30, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"] \arrow[dl, bend right, "s\llbracket-\rrbracket"'] \arrow[dr, phantom, "\overset{\mathsf{w}_t^{-1}}{\cong}"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) \arrow[dr, bend left, "s\llbracket-\rrbracket"] & \\
\mathrm{neuts}^{\mathrm{ps}}_{\Gamma}(\Gamma) \arrow[r] \arrow[dr, bend right, "s\llbracket-\rrbracket"'] \arrow[ur, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] \arrow[dr, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \textcolor{blue}{\overline{\mathrm{e}}}\llbracket\Gamma\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r, bend left=30, "\textcolor{blue}{\overline{\mathrm{e}}}\llbracket\Gamma\vdash t:A\rrbracket", ""{name=U, below}] \arrow[r, bend right=30, "\textcolor{blue}{\overline{\mathrm{e}}}\llbracket\Gamma\vdash t':A\rrbracket"', ""{name=D, above}] \arrow[dr, phantom, "\overset{\mathsf{w}_{t'}}{\cong}"] & \textcolor{blue}{\overline{\mathrm{e}}}\llbracket A\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r] \arrow[ur, phantom, "\overset{\overline{q}_A}{\cong}"] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \mathrm{norms}^{\mathrm{ps}}_{A}(\Gamma) \arrow[dl, bend left, "s\llbracket-\rrbracket"] \\
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend right=30, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) &
\arrow[Rightarrow, from=U, to=D, "\textcolor{blue}{\overline{\mathrm{e}}}\llbracket\tau\rrbracket"]
\end{tikzcd}$$

independent of τ!

$$\begin{tikzcd}
\mathrm{id}_{s\llbracket\Gamma\rrbracket} \in \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend left, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"] \arrow[r, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] \arrow[r, phantom, "\Downarrow s\llbracket\tau\rrbracket\circ(-)"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) \arrow[r, rightsquigarrow] & s\llbracket\tau\rrbracket
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend left, "s\llbracket\Gamma\vdash t:A\rrbracket\circ(-)"] \arrow[r, phantom, "\overset{\mathrm{w}_t^{-1}}{\cong}"] \arrow[dl, bend right, "s\llbracket-\rrbracket"'] \arrow[dl, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) \arrow[dr, bend left, "s\llbracket-\rrbracket"] \arrow[dr, phantom, "\overset{\overline{q}_A}{\cong}"] & \\
\mathrm{neuts}^{\mathrm{ps}}_{\Gamma}(\Gamma) \arrow[r] \arrow[dr, bend right, "s\llbracket-\rrbracket"'] \arrow[dr, phantom, "\overset{\overline{u}_\Gamma}{\cong}"] & \overline{\mathrm{e}}\llbracket\Gamma\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r, bend left, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t:A\rrbracket"] \arrow[r, bend right, "\overline{\mathrm{e}}\llbracket\Gamma\vdash t':A\rrbracket"'] \arrow[r, phantom, "\Downarrow \overline{\mathrm{e}}\llbracket\tau\rrbracket"] & \overline{\mathrm{e}}\llbracket A\rrbracket(\Gamma) \arrow[u] \arrow[d] \arrow[r] & \mathrm{norms}^{\mathrm{ps}}_{A}(\Gamma) \arrow[dl, bend left, "s\llbracket-\rrbracket"] \arrow[dl, phantom, "\overset{\overline{q}_A}{\cong}"] \\
 & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket\Gamma\rrbracket) \arrow[r, bend right, "s\llbracket\Gamma\vdash t':A\rrbracket\circ(-)"'] \arrow[r, phantom, "\overset{\mathrm{w}_{t'}}{\cong}"] & \mathcal{B}(s\llbracket\Gamma\rrbracket, s\llbracket A\rrbracket) &
\end{tikzcd}$$

independent of τ!

Proposition

For any cc-bicategory $\mathcal{B}$ *and interpretation* $s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathcal{B}$ *of base types, the interpretation* $s[\![\Gamma \vdash \tau : t \Rightarrow t' : A]\!]$ *depends only on* t *and* t'*.* $\square$

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$\rightsquigarrow$ For any derivable $\tau, \tau' : t \Rightarrow' : A$,

$$s[\![\Gamma \vdash \tau : t \Rightarrow t' : A]\!] = s[\![\Gamma \vdash \tau' : t \Rightarrow t' : A]\!]$$

Proposition

For any cc-bicategory $\mathcal{B}$ and interpretation $s : \mathsf{BaseTypes} \to \mathrm{Ob}\mathcal{B}$ of base types, the interpretation $s[\![\Gamma \vdash \tau : t \Rightarrow t' : A]\!]$ depends only on t and t'. ☐

$\rightsquigarrow$ For any derivable $\tau, \tau' : t \Rightarrow' : A$,

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Corollary

For any derivable $\tau, \tau' : t \Rightarrow' : A$,

$$\Gamma \vdash \tau \equiv \tau' : t \Rightarrow t' : A$$

is derivable. ☐

Coherence

$\rightsquigarrow$ cc-Bicategories are coherent

Coherence

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- $\rightsquigarrow$ "Suffices" to work in a CCC
  1. prove result in STLC
  2. $\beta\eta$-equalities $\rightsquigarrow$ 2-cells
  3. axioms guaranteed

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3. axioms guaranteed

Strategy

$\rightsquigarrow$ Coherence as a normalisation property

$\rightsquigarrow$ Normalisation proven semantically

$\rightsquigarrow$ If you work with universal properties enough

. . . higher-categorical proof builds on categorical proof

Future work: extend this to other structures

e.g. Kan extensions, monads, dependent products,. . .

Further reading

- *A type theory for cartesian closed bicategories*, LICS 2019
- *Relative full completeness for bicategorical cartesian closed structure*, FoSSaCS 2020
- *Cartesian closed bicategories: type theory and coherence*

cc-Bicategories

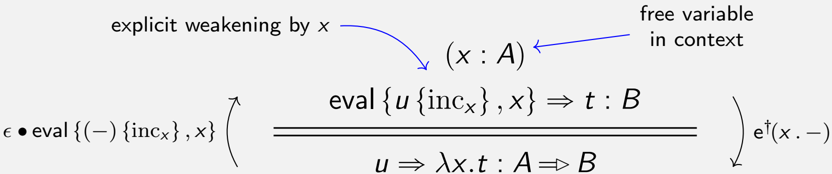
1-cells

$$\mathrm{eval}_{A,B} : (A \Rightarrow B) \times A \to B$$

Adjoint equivalences

$$\begin{tikzcd}
\mathcal{B}(X, A \Rightarrow B) \arrow[r, bend left, "\mathrm{eval}_{A,B} \circ (- \times A)"] \arrow[r, phantom, "\perp\simeq"] & \mathcal{B}(X \times A, B) \arrow[l, bend left, "\lambda"]
\end{tikzcd}$$

Rules for exponentials



explicit weakening by x

free variable in context

$$(x : A)$$

$$\epsilon \bullet \mathsf{eval}\left\{(-)\left\{\mathrm{inc}_x\right\}, x\right\} \;\Big\uparrow\; \begin{array}{c} \mathsf{eval}\left\{u\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B \\ \hline\hline u \Rightarrow \lambda x.t : A \mathrel{=\!\!\rhd} B \end{array} \;\Big\downarrow\; \mathsf{e}^{\dagger}(x \,.\, -)$$

$$\frac{}{f : A \mathrel{=\!\!\rhd} B, x : A \vdash \mathsf{eval}(f, x) : B}\ \mathsf{eval} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \mathrel{=\!\!\rhd} B}\ \mathsf{lam}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : A \vdash \epsilon_t : \mathsf{eval}\left\{(\lambda x.t)\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B}\ \epsilon\text{-intro}\ (\beta\text{-rule})$$

$$\frac{\begin{array}{c}\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A \mathrel{=\!\!\rhd} B \\ \Gamma, x : A \vdash \alpha : \mathsf{eval}\left\{u\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B\end{array}}{\Gamma \vdash \mathsf{e}^{\dagger}(x \,.\, \alpha) : u \Rightarrow \lambda x.t : A \mathrel{=\!\!\rhd} B}\ \mathsf{e}^{\dagger}(x \,.\, \alpha)\text{-intro}$$

+ three equational rules $\rightsquigarrow$ η-rule derivable