AN ESTIMATE OF THE RELATIONSHIP BETWEEN ZERO THRESHOLDS OF GAUSSIAN CURVATURE AND MEAN CURVATURE

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An estimate of the relationship between zero thresholds of Gaussian curvature and mean curvature

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Abstract

This paper tries to estimate and formulate the relationship between zero thresholds of Gaussian curvature and mean curvature in surface segmentation.

keywords: surface segmentation, zero threshold, Gaussian curvature, mean curvature.

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1. Introduction.

As the fundamental properties of surfaces [DeCA76], Gaussian curvature K and mean curvature H play an important role in surface perception, segmentation and recognition. By using the signs of curvatures K and H surfaces (or their points) can be classified exactly into eight types [BESL86].

However, the situation of surface classification in practice is not as good as in theory. As the surface curvatures are related to the second order derivatives, they are sensitive to noise, especially, for those curvatures around zero. How to treat the zero curvatures therefore becomes a critical problem. So far, the zero thresholding of curvatures has been discussed by many papers, where the thresholds are empirically imposed in the processing [BESL86][YANG88].

An empirical imposition might be feasible for single scale processing. However, this tactic is no longer appropriate for scale space processing [CAI87a,b,88a,b] due to the scale effects (including the reduction of noise, the distortion of surface and the changes of significant features at different scale levels). This suggests the complicated situation of the K and H thresholding in scale space [WITK83].

The thresholding problem could be addressed in two aspects. One aspect is how to give surface features an explicit scale-based thresholding, rather than an empirical imposition at individual scale level. [PONC87] have shown how to formulate the scale space behaviour of the curve primitives, such as, step, roof, smooth join, shoulder and bar, and use these results to set thresholds in edge detection. The other aspect is how to find possible inter-relationship between zero thresholds of different surface features, rather than to impose some isolated therefore perhaps casual values on different thresholds. This aspect becomes increasingly significant since the *combinations* of the signs of Gaussian and mean curvatures has been more and more frequently used in the surface segmentation. In this paper, we try to estimate and formulate the relationship between zero thresholds of Gaussian and mean curvatures so that once a threshold is fixed the other can be automatically produced by this formula.

2. An example of improper thresholding of zero K and H

There are nine combinations of the signs of K and H as shown below:

K \ H	−	o	+
−	saddle ridge	minimal	saddle valley
o	ridge	flat	valley
+	peak	(none)	pit

Figure 1. Surface shapes from the signs of K , H.

However, only eight of them can be used to classify surfaces. The rest one "K>0 and H=0" is an impossible case for surface type. From the definition of K and H in differential geometry, we have

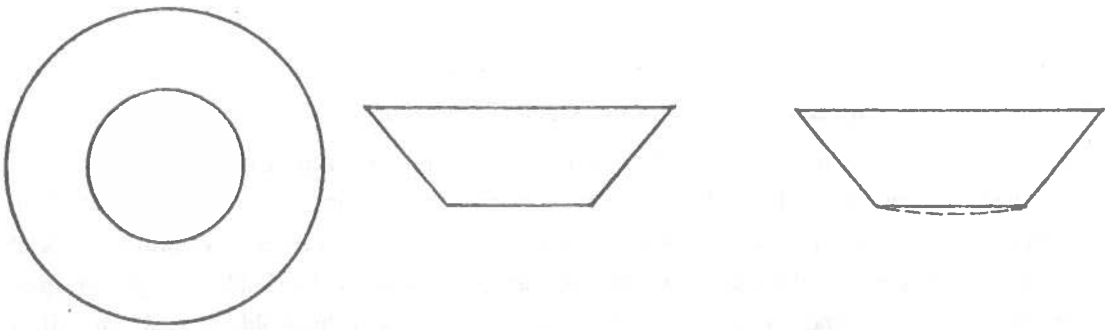
$$K = C_1 \cdot C_2$$

$$H = \frac{1}{2}(C_1 + C_2) \qquad (1)$$

where C_1 and C_2 are principal curvatures of surface. So, H=0 implies $C_1 = -C_2$, which leads to K<0. Thus, "K>0 and H=0" is a "phantom" type in theory.

Note that a strict zero is rarely met in computation. Small numbers will be seen as zero in computation if they are less than a given zero threshold. It is thus possible that an improper thresholding may therefore lead to the "phantom" case. When this happens, the surface is usually nearly flat. We now give a simulated example.

Suppose the given object is a plastic basin as shown in Figure 2.a. Its surface is composed of a conic wall patch and a planar bottom.



(a) (b)

Figure 2. A plastic basin and its shape under a weak perturbation.

The principal curvatures of the wall patch are 0.0 and 5.0, and both principal curvatures of the bottom patch are zero. According the KH sign classification the shape type of the wall will be "valley" and the shape type of the bottom patch will be "flat". The basin surface can be easily classified by setting small zero thresholds for K and H, say,

$$\varepsilon_K = \varepsilon_H = 10^{-3} \tag{2}$$

But when the bottom is under perturbation, its curvatures may change slightly as illustrated in Figure 2.b. For example, the principal curvatures now become $C_1 = C_2 = 9\times10^{-4}$. As both curvatures are close to zero, the bottom patch can still be classified as the flat type by a proper zero thresholding. From (1), the Gaussian curvature K of bottom patch will be much smaller than its mean curvature H, so it seems reasonable to get a planar patch by setting a zero K threshold which is much smaller than the zero H threshold, say,

$$\varepsilon_K = 10^{-7}\ , \qquad \varepsilon_H = 10^{-3} \tag{3}$$

However, this will be an improper thresholding. Although the shape of the wall patch can be correctly classified as a "valley" type, the shape type of the bottom patch we shall get is not a "flat" type "K=0 and H=0" but the "phantom" type "H=0 and K>0" because the curvatures of the wall patch satisfies

$$H = 9\times10^{-4} < \varepsilon_H\ , \qquad K = 8.1\times10^{-7} > \varepsilon_K \tag{4}$$

In fact, whenever $K > 0$, we have the following inequality:

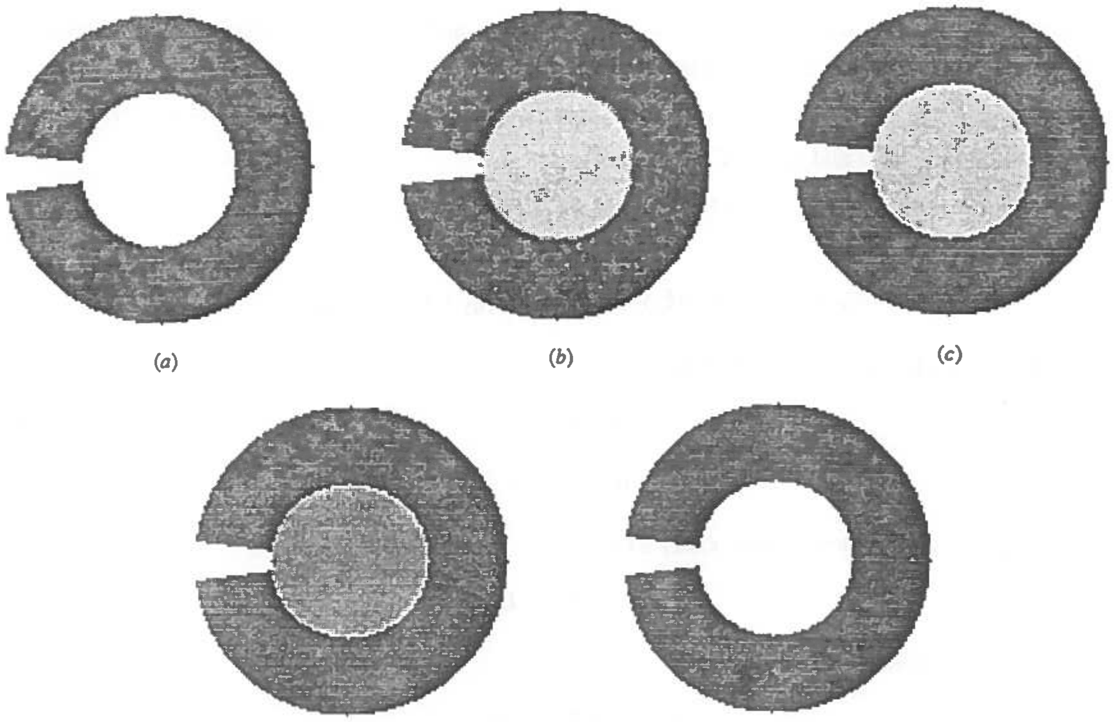
$$K = C_1 \cdot C_2 = |C_1| \cdot |C_2| \leq [\tfrac{1}{2}(|C_1|+|C_2|)]^2 = [\tfrac{1}{2}(C_1 + C_2)]^2 = H^2 \tag{5}$$

Hence, once a zero threshold ε_H is set up, the thresholds ε_K and ε_H should satisfy the inequality below to guard against the noisy perturbation around $K = 0$ and $H = 0$.

$$\varepsilon_K \geq \varepsilon_H^2 \tag{6}$$

Setting $\varepsilon_K \geq 10^{-6}$ corresponding to $\varepsilon_H = 10^{-3}$ may produce the right result as is expected.

The experimental results are shown as the KH sign images in Figure 3, where the whole background is a plane, the wall of the basin is a conic patch and the bottom of the basin is a plane under no perturbation or a shallow pit shape under a weak perturbation. For the convenience to compare the shape type of the basin bottom with that of the background in the KH sign image, a portion of the wall patch has been cut off.



(a) *(b)* *(c)*

(d) *(e)*

Figure 3. Experimental results of the zero thresholding for the basin surface classification.

Figure 3.a shows the correct classification result of the basin surface under no perturbation, where the shape type of the basin bottom is classified as the same to that of the background by using the thresholding (2). This is the result we want to get when the basin is under a weak perturbation. Figure 3.b shows the cosine-shading image of the basin surface under a weak perturbation, where the basin bottom is different from the background due to the shallow pit shape. Figure 3.c shows the shallow pit shape can be detected by using the thresholds:

$$\varepsilon_K = 8.5\times10^{-7}, \qquad \varepsilon_H = 10^{-3} \tag{7}$$

Figure 3.d shows the "phantom" classification result of the basin surface under a weak perturbation, where due to the improper thresholding (3) the basin bottom has the "phantom" shape type which is not only different from both the background and the wall but also different from the "pit" shape type shown in Figure 3.c. Figure 3.e shows the expected classification result of the basin surface under a weak perturbation, where by using the proper thresholding (6) the basin bottom returns to the same shape type to the background as shown in Figure 3.a.

It might be attractive to use thresholds as in (7) because they can partition surfaces and detect weak perturbations as well. Unfortunately, this is only possible for some very simple and simulated cases as the above example. In the presence of noise situation is more complicated for the real data of unknown objects, where a proper thresholding compatible with (6) will be meaningful and feasible to work.

3. Estimation to the relationship between zero thresholds ε_K and ε_H

The above inequality (6) is a brief formula, but it only considers the planar surface case and does not implies any effect of scales. There is a more complex and appropriate relationship between the zero thresholds ε_H and ε_K for all eight surface types.

Suppose that the principal curvatures have small perturbations ξ_1 in C_1 and ξ_2 in C_2. They introduce perturbations E_H in H and E_K in K respectively:

$$E_H = \frac{(C_1+\xi_1) + (C_2+\xi_2)}{2} - \frac{C_1+C_2}{2} = \frac{\xi_1 + \xi_2}{2} \tag{8}$$

$$E_K = (C_1+\xi_1)(C_2+\xi_2) - C_1C_2 = (C_1\xi_2 + C_2\xi_1) + \xi_1\xi_2 \tag{9}$$

For simplicity, let $\xi_1 = \xi_2 = \xi$, therefore

$$E_H = \xi \tag{10}$$

$$E_K = (C_1+C_2)\xi + \xi^2 \tag{11}$$

Thus, E_K and E_H are now related via H as below †:

$$E_K = 2H\cdot E_H + E_H^2 \tag{12}$$

Set $|E_H| = \varepsilon_H$, then

$$|E_K| \leq 2|H|\cdot\varepsilon_H + \varepsilon_H^2 \tag{13}$$

Hence, to guard against the perturbation, the zero threshold of the Gaussian curvature K should satisfy the following inequality:

$$\varepsilon_K \geq sup|E_K| = 2|H|\cdot\varepsilon_H + \varepsilon_H^2 \tag{14}$$

Otherwise, for instance, an exact $K = 0$ plus the perturbation E_K will be beyond the zero band $[-\varepsilon_K , \varepsilon_K]$, thus may lead to an incorrect surface classification.

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† This result implies that the perturbation introduced in K might have a larger amplitude than that in H when $H>\frac{1}{2}$, and a much larger amplitude when $H>5$.

Comparing with (6), there is a term containing H in (14), which introduces the scale effects via the mean curvature H changing over scales. By (14), the value of ε_K can be automatically derived from ε_H. If the value of ε_K is set for the whole surface S, we may have

$$\varepsilon_K = \varepsilon_H^2 + 2 \max_S |H| \cdot \varepsilon_H \tag{15}$$

Where, it is no surprise that $\varepsilon_K >> \varepsilon_H$ when $\max_S |H| > 5$. As for the case $H \equiv 0$, we get $\varepsilon_K \geq \varepsilon_H^2$, just as in (6). Hence, we can use (15) to explain why people impose a zero threshold on Gaussian curvature which is usually larger than the zero threshold of mean curvature. For example, [BESL86] once used $\varepsilon_H = 0.015$ and $\varepsilon_K = 0.06$ as the zero thresholds of Gaussian curvature and mean curvature.

In scale space, when the smoothing scale increases, the surface becomes more and more smooth. Note that the high curvature points on a smooth surface are always in a small number even though including some false high curvature points produced by noise. Also note that the aim of setting ε_K and ε_H is to properly treat those zero curvature points. Setting H an average value over the whole surface may be feasible in the scale space processing. The following formula is thus suggested:

$$\varepsilon_K = \varepsilon_H^2 + 2 \, \underset{S}{Average} |H| \cdot \varepsilon_H \tag{16}$$

The experimental results of surface segmentation using this formula in scale space processing can be found in [CAI88a].

4. Summary

This paper formulates the relationship of zero thresholds of Gaussian curvature and mean curvature. Once a proper zero threshold ε_H has been chosen, the zero threshold ε_K will be given automatically by this formula, which is particularly convenient for scale space processing, where any zero threshold empirically imposed on K and H at a certain scale is no longer appropriate at other scales.

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